A Markov Chain Interpretation of some Arpeggios in Beethoven's Moonlight Sonata, 1st Movement

Hisashi Kobayashi[1]

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Abstract: We present a *Markov chain* interpretation of some arpeggios in the first movement of Beethoven's Moonlight Sonata, which is composed in a ternary form, i.e., A-B-A' form.

In Section II, we analyze the arpeggios of Part A. Expert pianists may wish to skip this section. Our analysis, however, should have a pedagogic value to novice pianists, including this author, in helping them identify the chords and memorize this famous sonata more easily.

In Section III, we propose to apply a Markov chain representations of the main arpeggios in section B. We show that the passages of the 32nd through 37th measures, which are arpeggiated from D# diminished 7th, C# minor, C# diminished 7th, and F# diminished 7th chords, respectively, can be concisely presented in terms of "state transition diagrams," often used for analysis of a Markov chain in probability theory [4]. This representation leads to simple *algorithmic* descriptions of the arpeggios. Perhaps this simplicity is a key to the beauty of this part of the sonata.

In Section IV, we present harmonic analysis for Part A'. Although there are a few entirely new passages, a large portion of Part A' is a recapitulation of the passages of Part A, either as straightforward repetitions or in transposed forms.

We hope that our chord analysis will be of some value to mathematically inclined pianists, and that the transition diagram representation will provoke interest among music theorists, possibly as a new tool to augment such existing tools as the Schenkerian graph and notation [5,6,7].

I. Introduction

Sonata Number 14, Op. 27, No.2 was composed by Ludwig Van Beethoven (1770-1827) in 1801, with the title "Sonata Quasi Una Fantasia (almost fantasy)," and dedicated to Countess Julietta (Julia) Guicciardi (1782-1856), a young student of Beethoven [1]. Sonata Number 14, better known as the Moonlight Sonata, is Beethoven's most recognizable sonata and is arguably one of the most significant pieces of classical music ever written [1, 2].

This piece became well known, even in his own day. Beethoven reportedly got annoyed that this sonata became so popular that it might eclipse his other work [1]. In 1832, five years after Beethoven's death, the famous German poet Ludwig Rellstab (1799-1860)[2], listening to the first

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[1] For any comments and questions, please send them to the author's email address is Hisashi at Princeton.EDU

[2] For example, the relic of Franz Schubert's Ständchen (Serenade) was written by Rellstab.

movement, Adagio sostenuto (slowly in a sustained style), envisioned a boat on the moonlit water of Lake Lucerne, and gave the name “Moonlight” (“Mondscheinsonate” in German) to this sonata [1].

Music historians argue, however, that this sonata has nothing to do with Beethoven’s love for Julia, nor Rellstab’s description. The origin of the motif of the sonata, characterized by its ostinato triplets should be ascribed to the same ostinato triplets Mozart used for the death throes of the Commendatore who was stabbed by Don Giovanni: see e.g. Daniel Barenboim’s video “5 Minutes on Beethoven-The Moonlight Sonata (C# minor).” In a similar vein, it will be worth noting that Frédéric Chopin did not publish “Fantaisie Imprompu” (also in C# minor) while he was alive. Some music historian conjectured its similarities to the third movements of Beethoven’s Moonlight sonata might have been the reason. See e.g., https://en.wikipedia.org/wiki/Fantaisie-Impromptu

The first movement in C# minor is written in the A-B-A’ form. It consists of 69 **measures** [3] (see the music sheets attached at the end). We identify Part A as measures 1 through 27; Part B, measures 28 – 40; and Part A’, measures 41-69 (see the music sheet https://musescore.com/nicolas/scores/30321 attached at the end of this article).

The term “arpeggio” is defined as a sequence of tones taken from a certain chord or harmony, and the verb “arpeggiate” is used to express this process. In this report we use the term arpeggio synonymously with its associated chord, when it is unambiguous. We use abbreviations and often drop definite and indefinite articles, so as to be concise. For instance, we may write “an A-octave” as “A-8ve”, “a G# major 7th chord” as “G# 7”, “measure 1” as “m.1”, etc. When we use a roman numeral of a chord, we distinguish its root position, first inversion and second inversion by adding “a”, “b” or “c” at the numeral chord. The first inversion of a dominant chord V will be written as Vb.

II. Arpeggios in Part A of the First Movement

The first movement opens with an octave in the left-hand (LH) and a triplet figuration in the right. The right-hand (RH) in m.1 and m.2 repeats eight times the same arpeggio from C# minor in 2nd inversion, which forms an *ostinato*[4] triplet rhythm [2], and dominates throughout Part A. The LH of m.1 is C#-8ve, whereas in m.2, it descends to B-8ve**,** we consider it as a *passing note* [3], which further descends to A-8ve and then to F#-8ve in m.3, where the RH plays an arpeggio of A-major in the root position[5], followed by D-major in the 2nd inversion.

[3] A “measure” is synonymous with a “**bar**,” which seems a preferred term in the British literature on music.
[4] Ostinato means “stubborn” or “persistent.” Here an ostinato triplet rhythm means that the same triplet appears repeatedly. See for details https://en.wikipedia.org/wiki/Ostinato
[5] More precisely, the root position of the **submediant** major chord (i.e., **VIa**) in C# minor, where the roman numeral **VI** represents the degree from the tonic note (C# in this case), and the lower case letters **a, b, c**, …, are a short-hand notation for **root form**, **1st inversion**, **2nd inversion**, etc. [3]..

M.4's (or Measure 4's) 1st beat G#-B#-F# is G# dominant 7th in root position[6] , although its 5th degree D# is absent, The 2nd beat returns to C# minor in 2nd inversion, which appeared already in m.1 and m.2 (but with different octaves in the bass). The 3rd beat is G# suspended 4 (or G# sus4 [7] for short) and the 4th beat is again from G# dominant 7th in root position (or V7a in this C# minor *key area*) , although F# does not appear.

M.5 resembles m.1, although the LH plays C#-G#-C# instead of C# 8ve. The 1st beat is the 1st inversion of C# minor, the 2nd and 3rd beats are exactly the same as the 8 beats of m.1 and m.2. A *melody* note is introduced on top of the rhythmic beats, with a top note G# [8] to be played by RH at the beginning of the 4th beat, and immediately after the 4th beat, followed by a long note of the same G# at the beginning of m.6. This melody appears repeatedly throughout the first movement (see measures 6-7, 10-11, 16-17, 23-24 , 24-25, 42-43, 43-44, 46-47, and 47-48). Together with the ostinato triplet rhythm, this melodic structure determines an overall tone of Parts A and A'. Towards the end of Part A', however, the melody note is played by the LH (see measures 60-61, 61-62, 62-63 ,63-64 , 64-65 , and 65-66), where a quarter note of base G# is added to the 4th beat, and its sixteenth note immediately after the 4th beat, followed by the base G#-C, G#-C#, or G#-G#.

M.6 expands the G# dominant 7th introduced in m.4, and the RH repeats the triplet (G#-D#-F#). The base is B#-G#-B# instead of G#-8ve. The melodic tone, introduced in the last beat of m.5, is repeated in the 4th beat of m.6 and the first half of m.7.

The first half of m.7 returns to C# minor, as in m.1, with a melodic note of G4# added. The second half introduces F# minor (in 1st inversion) with F#-8ve base, and a melodic note of A4 is added. After m.7, arpeggios from major chords preside over the next 5 measures. M.8 is E major in 1st inversion, followed by B major 7th. The melodic notes of A4, F4# and B4 are added to beats 1&2, 3 and 4, respectively.

We can see that m.1 though m.2 have stayed in the same key as the key signature of this piece, i.e., C# minor. However, at m.9, we observe that the music temporarily digresses to another key, E-major. The last chord of m.8 is B-major 7th, i.e., VII7 in C#. This chord, called a *pivotal*-chord, is V7, i.e., the dominant 7th in E major. We say that m.9 (and m.10, as well) is in the *key area* of this temporary key E-major. This digression of the key is referred to as a *modulation* [3] in music theory. We also note that the music comes to a temporary rest at the first note of m.9. This progression from a dominant tonic chord (in the new key E-major) is referred to as a *perfect cadence*.

M.9 introduces a new ostinato triplet rhythm (G#-B-E) , with a melody note E4, which is lower than any of the preceding ones: G4#, F4#, B4. The bass (LH) is raised to E-8ve. M.10 changes to

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[6] More precisely, the root position of the dominant 7th chord (i.e., V^7a) in C# minor [3].

[7] A suspended chord, or sus chord for short, is a chord in which the third (major or third) is omitted, replaced by either a perfect 4th or a major second; the former is called a sus4 chord, whereas the latter, a sus2 chord.

[8] To be exact G4#, where G4 means G in the 4th Octave. It is the American convention to denote the "middle C" as C4.

E minor. In m.10 and m.11, the "melody featuring dotted rhythm" introduced in m 5 and m.6 is resumed. The triplet rhythm of the RH is, in m.10 and m.11, (G#-B-E) and (G-B-E), respectively.

If we interpret that key area of the music shift from E major to C major in m.11, the harmonic chord of m.11 is a G *dominant* 7^{th} chord (G-B-D-F), i.e. V7 in C major. In this modulation the pivotal chord is E-minor in 1^{st} inversion (GBE), i.e., a mediant chord, iiib, in the key area C-major.

M.12 consists of four different chords: C-major in 2^{nd} inversion (G-C-E), E minor in 1^{st} inversion (G-B-E), G-diminished 7^{th} (G-A#-C#-E) and F# 7^{th} suspend (F#-C#-E). The bass part descends from E-8ve (m.10) to D-8ve (m.11), and down to C-8ve, B-8ve and A#-8ve (m.12).

The first half of m.13 through m.15 are B minor but in different forms; the 2^{nd} inversion (F#-B-D) in m 13 and m.14, and the root form (B-D-F#) in m. 15. The second half in these measures are E-minor 6^{th} (m. 13), F# major (m.14) and B major (m.15).

The first note of m.15 provides a second *cadence* in this music, which now modulates to B minor chord, with a pivotal chord F# major, which is V in the B minor key. Thus, this cadence is also a perfect cadence.

M.16 has as rhythmic beats a triplet (B-E-G), which is an E minor in 2^{nd} inversion, with the dissonant melody note B4# played throughout the first three beats and another dissonant note A4# in the 4^{th} beat. The LH moves upward from B-8ve, E-8ve to G-8ve, and then down to E-8ve.

M. 17 returns to the second half of m.15 (B major) with the ostinato triplet rhythm of (B-D#-F#). The melody note B4 is added to 1^{st} - 3^{rd} beats, and again to 4^{th}. M.18 is an exact repeat of m.16.

The first half of m.19 is identical to m.17, including the B-8ve base. In the second half, G# diminished 7th, the 2^{nd} - 3^{rd} notes (D-E#) of the triplets are shifted down a semitone (a minor 2^{nd}) from those (D#-F#) of the first half, which in turn are shifted down a semitone from those (E-G) in m.16 and m.18.

The 1^{st} half of m.20 has (B-C#-G#) as a triplet rhythm, with the LH base F# 8ve, and B4 as a melody note. This passage sounds quite different from any previous phrase. This is from a C# major 7^{th} chord. The 2^{nd} half, F# minor, already appeared in 2^{nd} half of m.7.

M.21 introduces G major (G-B-D) for the first time here, although its extended version G major 7^{th} already appeared in m.11. The 2^{nd} half of m.21 introduces B# diminished 7^{th} (B#-D#-F#-A)[9]. Diminished 7^{th} chords play a major role in Part B, as discussed in Section III.

[9] A diminished 7^{th} chord is built on the 7^{th} step or degree of a major or minor scale. It is composed of three minor thirds stacked on top of one another. When a diminished 7^{th} chord is inverted, the result is **equivalent** to another diminished 7^{th} chord [3]. F# dim 7 and D# dim 7 can be viewed as the subdominant dim 7 and supertonic dim 7 in C# minor.

The 1st half of m.22 is from F# minor in 2nd inversion (C#-F#-A). This chord appeared already in m.7 & m.20, but in root form (F#-A-C#). The 3rd beat of m.22 is from C#-suspended 4th, and the 4th beat is from C#-major, which is also new, although C#-major 7th already appeared in m.20.

A good portion of m.23 through m.25 is obtained by transposing up by perfect 4th (i.e., 2 ½ steps) the passage in m.5 through m.7. Thus, m.23 is F# minor, m.24 is C# major 7th and the first half of m.25 is F# minor. The bass part also moves up from C#-G#-C# and C-G#-C to F#-C#-F# and F-C#-F, respectively. There is a difference, however, in that in m.5, the first beat is C#-minor in 1st inversion, whereas in m.23, the 1st beat is F# minor in root form. Consequently, the base F#-C#-F# in m.23 is set one octave below compared with what would be at its comparable position. Note that the first note of m.23 creates a third cadence in this music. The ending beat of m.22 is C# major chord, which is a V chord in the key area of F# minor that begins in m.23. Hence this is another perfect cadence.

The second half of m.25 is D# dim 7, followed by F# minor. M.26 is from a G# 7th chord, which earlier appeared in m.4 and m.6, but in different inversions.

The first half of m.27 is C# minor in 1st inversion. C# minor appeared several times already (i.e., measures 1, 2, 4, 5, & 7), but they were all in 2nd inversion. The third beat is F# dim 7th, and the 4th beat is G dim 7th.

III. Part B: A Markov Chain Representation

Note that in all measures of Part A, the three tones within each beat always went up. Arpeggios with these rhythmic beats, which some musicians describe aptly as a “hypnotic” set of arpeggios, however, end at the end of the 108th beat (27 x 4=108). Note also that a dominant pedal tone, G#-8ve, begins in m.28, and is sustained over the next 14 measures (from m.28 to m.41) , the only exceptions being in the second half of m.40 and the first half of m.41. Despite of these noteworthy major changes that begin at m.28, one might argue that Part B section has commenced at m.23, immediately after the third cadence.

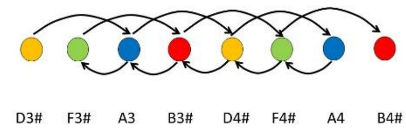
The key area F# minor will last till the end of m.41, where the fourth cadence appears at the beginning of m.42, and the key area changes back to the signature key C# minor.

In m.28 and m.29, which are basically both G# major 7th , each triplet takes the form “*M*-C4-D4#,” where *M* is a *melodic* tone *variable*, and in m.28 the sequence of *M* is (B#5, G4#, A4, F4#), whereas it is (𝄾, G3#, A3, F3#) in m.29, where “𝄾” is an eighth rest note. The melody tones in m.29 are one octave lower than m.28, an exception being the rest note in the 1st triplet of m.29 . The melodic notes A3 and A4 are non-chordal notes and create a tension against the chord harmony. Since this is a V7 chord in the key area of C# minor, it is a G# dominant 7th chord.

M. 30 & m.31, which are both C# minor chords, including the G#-8ve base part, are analogous to m.28 & m.29. Each triplet takes the form “*M*-E-G#,” where the melody tone *M* is also from a C#-minor (C#-E-G#). In m.30, the sequence of the variable *M* is (E3, C5#, E5, C4#), whereas in m.31, it is (𝄾, C4#, E4, C4#) , one octave lower version of m.30.

The arpeggio of m.32 consists of notes from D# dim 7th (D#, F#, A, B#) . Note that F# dim 7, which appeared earlier in m.21 and m.25, is the 1st inversion of D# dim 7. In Fig. 1, we depict this arpeggio in terms of a *state transition diagram*, where the term "state" here simply refers to one of the piano keys used in a given arpeggio.[10] Each state is represented by a circle and a transition from a state to the next state is represented by an arrow or arc.

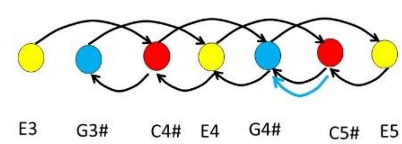
Figure 1. Arpeggio in measure 32



In Fig. 1, we use "orange", "green", "dark blue", and "red" circles to identify D#, F#, A and B# keys, respectively. Starting from "D3#", the arpeggio jumps two step**s**[11] to the right (i.e., to A3"), and then retreat one step to the left (i.e., to "F3#"). By repeating five times this simple cycle [12] of "jump two steps forward, and one step backward", the arpeggio reaches "F4#". After finishing the first half of the 6th cycle (i.e., "jumping two steps to the right"), this arpeggio will arrive at B4#, where it ends.

The arpeggio of m.33 is composed of notes in C# minor chord (C#, E, G#). Fig. 2 is the transition diagram[13] of this arpeggio. Here "yellow," "blue," and "red" circles represent "E", "G#" and "C#" keys, respectively.

Figure 2. Arpeggio in measure 33



Starting from "E3," it follows the same *algorithm* as in the arpeggio of Fig. 1, i.e., "Jump two steps to the right, and retreat one step to the left." After repeating this cycle again five times, the arpeggio reaches "C5#." In the first half the sixth cycle, the arpeggios should jump to G5# (which should be on the right of E5), if the above algorithm were faithfully followed.

Beethoven somehow decided to retreat to "G4#" as shown by a blue arrow. An obvious alternative is to let this arpeggio end at G5#, not at G4#. Shouldn't it sound equally beautiful, and maybe even better, because of its consistency with the other arpeggios? The author conjectures that Beethoven probably wanted to have this arpeggio contained within two octaves. This would make the arpeggio somewhat more "reserved" than letting it go up to G5#. The

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[10] A state transition diagram is often used in the analysis of a well studied random process, a *Markov process* [4], and also in *Automata* theory. https://en.wikipedia.org/wiki/Automata_theory

[11] The term "step" should not be confused as the music term "step" as used in "a whole step", "a half step," etc.

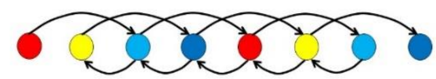
[12] Note that a cycle in the state transition diagram and a beat or triplet are not synchronous. The 2nd cycle and the first half of the 3rd cycle takes (C4-A4-D4#), which is the 2nd beat of the 32nd measure.

[13] We depict in Figure 2 that the distances between the states or key positions of C# minor are not equal: the distances between E and G# , between G# and C#, and between C# and E are 2 whole steps, 2 ½ steps, and 1 ½ steps, respectively (2+ 2 ½ + 1 ½ =6).

author believes that this kind of question is an excellent example in which a state transition diagram representation leads to important questions concerning possible choices in music composition.

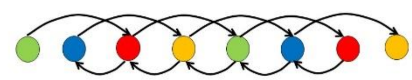
The arpeggio of m.34 is C# dim 7, (C#, E, G, A#), which appeared in the last beat of m.27. Fig. 3 is the transition diagram of this arpeggio. Here "red," "yellow," "light blue" and "blue" represent the C#, E, G, A# keys, respectively. The pattern not only looks the same as Fig. 1, but also sounds melodically identical. This is because this arpeggio (C# dim 7) is obtained by transposing up the arpeggio of Fig. 1 (D# dim 7) by minor 6th (i.e., 5 steps).

Figure 3. Arpeggio in measure 34



Now we analyze a long arpeggio that runs from m.35 to m.37. This arpeggio is based on F# dim 7 chord, and its transition diagram is shown in Fig. 4, which readily explains why this arpeggio is melodically identical to that of Fig. 3. F# dim 7 is obtained by transposing up C# dim 7 by perfect 4th (i.e., 2 ½ steps). We earlier noted that F# dim 7 is a 1st inversion of D# dim 7. It should be also clear that Fig. 4 is a 7 ½ steps[14] lifted version of Fig. 1.

Figure 4. Arpeggio in measure 35



Since these four arpeggios are all governed by the same simple algorithm, and they are based on dim 7th chords (except the one in Fig. 2), these three arpeggios are melodically equivalent to each other. Note that the *distances* or *intervasl* between any adjacent "states" in these diagrams (Figs. 1, 3 and 4) are equal to a minor 3rd (i.e., a whole tone & one semitone). Note also that minor 3rd x 4 = one 8ve. In Figures 1, 3, and 4, we depict these circles (or piano key positions) to be equally apart, and they form an arithmetic series [15] or progression. The circles or piano keys in Fig. 2, however, do not possess this nice property.

The starting point of m.36 is "B5#," which is the key where the arpeggio of the previous measure should be at, if the second half of the above algorithm is executed beyond the boundary of the bar between m.35 and m.36. At B5#, Beethoven chose to apply the same "algorithm" in the reverse direction, i.e., "Jump two steps to the left, and retreat one step to the right," as depicted in Fig. 5. At the end of m.36, the arpeggio has descended to "D4#." This descent continues, well beyond the bar boundary, into m.37, 3rd triplet, reaching "F3#" in Fig. 5. The 4th triplet C#-F#-A

[14] If we ignore the octave, the arpeggio of Figure 4 is a minor 3rd lifted version of that of Figure 1, because 7 ½ = 1 ½ (modulo 6)

[15] Note that an arithmetic series in terms of the interval between the states or piano keys correspond to a *geometric series* or progression in terms of the sound frequency of the acoustic tones.

is F#-minor, which appeared already several times in various forms (i.e., measures 7, 20, 22, 23, and 25).

The first half of m.38, m.39 and m.40 are exactly the same, and they are basically the dominant G# 7th chord, with a dissonant A creating a tension. The 3rd triplet of m.38 is D# dim, whereas that of m.39 is D major. The last triplets of both measures is F# minor, which appeared in m.37, in 2nd inversion form.

The second half of m.40 is A-major chord.

The first half of m.41 F# minor 6th chord. The 3rd beat is G#-major (see the 1st half of m.28 and m.29). The 4th beat is G# dominant 7th in C#-minor (see m. 4, 4th beat; m.28 & m.29, 4th beat).

The music arrives at a fourth perfect cadence at the first note of m.42.

IV. Arpeggios in Part A' of the First Movement

The passage of m.42 to m.45 is a complete repeat of m.5 through m8, (A-G#-F#) and the first note of m.46 provides the 5th perfect cadence of this piece.

M.46 differs from m.9 in that the 2nd to 4th triplets are 1st inversion of an E-major chord, while in m.9 they retain the root form of 1st triplet.

The passage from m.46 to m.48 sound very familiar. Indeed, if the notes in m.5 to m.7 are transposed up a minor 3rd, we obtain this passage. This passage was also in m. 41 to m.43, with proper modification in the base parts: the LH in m.41 and m.42 are E-8ve & D# 8ve, instead of E-B-E & D#-B-D#, respectively. M.48's 3rd and 4th triplets are G# 7th in its 1st inversion, and C# minor in root form.

The first half of m.49, a G# 7th chord in 2nd inversion, is identical to that of m.26. The second half of m.49, a C# minor chord in 1st inversion, is identical to the first half of m.27. M.50 is a D major in root form, followed by a G# dominant 7th in the key area of C# minor.

The first tone of m.51 defines the sixth perfect cadence of this music.

The next four and half measures (m.51- m.55's first half) is obtained by transposing up a tone (i.e., a major 2nd) the passage from m.15 – m.19's first half, with the only difference being that the first three beats of m.51 has C5# as a melody tone on top. M.51 consists of C# minor in root form, followed by C# major in root form. This structure is parallel to that of m.15, where B minor is followed by B major.

In the latter half of m.55, their 2nd & 3rd notes (F#, A) are a semitone higher than the corresponding notes (F, G#) in the first half of m.55. This is in sharp contrast of m.19, where the notes (D#, F#) was lowered to (D,E#).

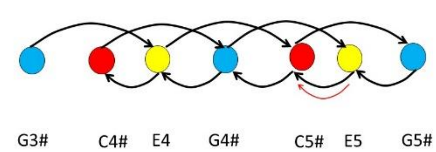
The 1st -3rd beats of m.56 is a B major 7th chord, and its transposed versions appeared already a few times: m.6 (G# major 7th), m.24 (C# major 7th), and m.43 (G# major 7th) . The fourth beat of m.56 is an E major chord.

The arpeggio of m.57 is new and not repeated elsewhere. M.58 is also unique and interesting; each beat takes the form (*M*-C#-D#), where *M* changes from F#-octave, G#-8ve, and up to A-8ve, whereas the bass part descends from A-8ve, G#-8ve and to F#-8ve.

M.59's first half is C# minor, and its second half is G# 7th in 3rd inversion. A G# 7th arpeggio appeared several times earlier, but they are all in different inversions: m.4 (in root form, but D# does not appear anywhere), m.6 & m.43 (in root form, but B# appears only in the base), m.26 (2nd inversion, B# is in the base), m.48 (1st inversion, D# in the base), m.49 (2nd inversion, D# is in the base), and m.50 (1st inversion, but D# does not appear anywhere).

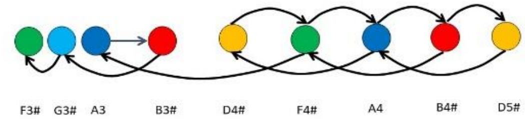
The passage of m.60 & m.61 is a variation of m.5 & m.6; the melodic note of G2# is played by the LH during and immediately after the last triplet of m.60 and the base G2# - B2# of three beats long at the beginning of m.61.

Figure 6. Arpeggio in measure 62



The arpeggio of m.62, depicted in Fig. 6, is similar to that of m.33 of Fig. 2. It is also arpeggiated from C# minor chord, and is governed by the same simple algorithm, i.e., jump two steps to the right, followed by one step back. The difference is that it begins with G3#, namely, it is one cycle behind that of Fig. 2, which starts with E3 and enters G3# after the first cycle. Since the chord is not a diminished 7th chord, the melody of this passage is not melodically equivalent to that of Fig. 2, but the first four cycles (out of 5 ½ cycles) in the measure is the same melody. Beethoven lets the arpeggio end at C5# instead of C6#as depicted by a red arrow in Fig. 6, which is consistent with the decision he made for the arpeggio of Fig. 2, where the arpeggio ends at G4# instead of G5#.

Figure 7. Arpeggio in measure 63



The arpeggio of m.63, shown in Fig. 7, resembles that of Fig. 5. The first part of the arpeggio is governed by the following algorithm: "Take one step to the right, and jump two steps to the left," which is equivalent to switching between the first and second halves of a cycle in the descending algorithm that governs Fig. 5. Starting from state B4# (the second circle in red from the right) which is in B#

diminished 7^{th} [16] the arpeggio goes to D5# and then to A4 in the first cycle. After three and half cycles, it arrives at F4# (the second visit). Then it jumps three steps to the left and enters A3 instead of jumping to B4#, a consequence of a digression from the algorithm. This note A3 can be viewed as a passing note that connects the arpeggio from B# diminished 7^{th} chord to an arpeggio of the last beat (A3/B4# -G3#-F3#), which corresponds to G# 7^{th} (b9) chord (G#, B#, D#, F#, A). Recall that this chord appeared in m.28. In m.63, 4^{th} beat, D# does not appear. Thus, this arpeggio can be alternatively ascribed to F# dim 9 chord, which includes (F#, G#, A, B#, E).

M.64, as depicted in Fig. 8, is almost identical to m.62 (Fig. 6) except that the first note is E3-C4#, instead of G3#.

It is evident that m.65 is identical to m.63, shown in Fig. 7.

V. Concluding Remarks

There must be numerous books and articles which report on analysis of this famous sonata. To the best of the present author's knowledge, however, an application of state transition diagrams to arpeggios has not been reported.

There is nothing probabilistic or stochastic in music once composed in a final form, but a state transition representation of the music as a Markov chain or automaton will provide a useful way of understanding the structure of music. It may suggest viable alternative options to consider in its composition stage, as remarked in our discussion of the arpeggio of measure 33 (Fig. 2).

It may certainly help a performer of jazz music, for instance, improvise cleverly.

Our future study is to relate this approach to the so-called Schenkerian school's analysis of Moonlight sonata [7] and other music. Another interesting avenue for further investigation is to relate the state transition representation to *music set theory*, suggested by the violinist Ray Iwazumi.

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[1] *Beethoven's Moonlight Sonata*, Transcribed & Performed by Craig Stevens, Santorella Publication.

[16] As noted earlier A dim 7, C dim 7, D# dim 7 and F# dim 7 are equivalent in that they consists of four notes A, C, D# and F#.

[2] *Piano Sonata No. 14 (Beethoven)*-Wikipedia, https://en.wikipedia.org/wiki/Piano_Sonata_No._14_(Beethoven)

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[4] Hisashi Kobayashi, Brian L. Mark and William Turin, *Probability, Random Processes and Statistical Analysis*, Cambridge University Press, 2012.

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