No. 9: Application of the Euler-Maclaurin summation to log-differentials of $M(t) = |\zeta(\frac{1}{2} + it)|$

Towards a Proof of the Riemann Hypothesis

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Abstract

This is our first report on numerical evaluation methods for the Riemann zeta function and its related functions. After reviewing the classical Euler-Maclaurin summation, we follow Edwards [2] in the use of this summation formula to evaluate the Riemann zeta function. Then, we extend this method to the evaluation of log-differentials of the modulus $M(t) = |\zeta(\frac{1}{2} + it)|$. We are interested in the function $M(t)$, since it is closely related to the function $\Xi(t) = \zeta(\frac{1}{2} + it)$ and Hardy's Z-function, which we have investigated in our previous reports [5, 6, 7, 8].

Key words: Euler-Maclaurin summation, Bernoulli polynomials, Bernoulli numbers, Riemann zeta function $\zeta(s)$, Modulus function $M(t) = |\zeta(\frac{1}{2}+it)|$, Function $\Xi(t) = \xi(\frac{1}{2}+it)$, Z-function $Z(t)$, Riemann-Siegel theta function $\vartheta(t)$, log-differentials.

1 Introduction: Euler-Maclaurin summation, Bernoulli polynomials, and Bernoulli numbers

The material that we present in this section draws heavily upon Edwards [2], Chapter 6, who gives a comprehensive account of this topi. For a continuously differentiable function $f(x)$ defined for $x \in [M, N]$, we can show that the following formula holds ([2] p. 99, (3)):

$$\sum_{n=M}^{N} f(n) = \int_M^N f(x)\,dx + \tfrac{1}{2}[f(M) + f(N)] + \int_M^N (x - [x] - \tfrac{1}{2}) f'(x)\,dx, \tag{1}$$

where $[x]$ is the floor function, i.e., $[x]$ is the largest integer that does not exceed x. The third term can be manipulated by using the periodic property of $x - [x] - \frac{1}{2}$, and by using integration by parts:

$$\begin{aligned}
\int_M^N (x - [x] - \tfrac{1}{2}) f'(x)\,dx &= \sum_{n=M}^{N-1} \int_n^{n+1} (x - [x] - \tfrac{1}{2}) f'(x)\,dx = \sum_{n=M}^{N-1} \int_0^1 (t - \tfrac{1}{2}) f'(n+t)\,dt \\
&= \sum_{n=M}^{N-1} \left[(t - \tfrac{1}{2}) f(n+t) \Big|_0^1 - \int_0^1 f(n+t)\,dt \right] \\
&= \sum_{n=M}^{N-1} \left[\tfrac{1}{2} f(n+1) + \tfrac{1}{2} f(n) \right] - \sum_{n=M}^{N-1} \int_0^1 f(n+t)\,dt \\
&= \tfrac{1}{2} f(M) + f(M+1) + \cdots f(N-1) + \tfrac{1}{2} f(N) - \int_M^N f(x)\,dx, \qquad (2)
\end{aligned}$$

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which indeed proves the formula (1). This formula can be transformed into a useful form by further use of integration by parts, and by adopting the Bernoulli polynomials, as will be shown below.

The nth Bernoulli polynomials $B_n(x)$ is the unique polynomial of degree n with the property

$$\int_x^{x+1} B_n(t)\,dt = x^n. \tag{3}$$

By differentiating the above, we obtain

$$B_n(x+1) - B_n(x) = \int_x^{x+1} B_n'(t)\,dt = nx^{n-1}. \tag{4}$$

The last equality implies that $\dfrac{B_n'(x)}{n}$ satisfies the definition of $B_{n-1}(x)$, which is the integrand in (3), with n replaced by $n-1$. Hence,

$$B_n'(x) = nB_{n-1}(x). \tag{5}$$

The first four Bernoulli polynomials are given as

$$B_0(x) = 1, \quad B_1(x) = x - \tfrac{1}{2}, \quad B_2(x) = x^2 - x + \tfrac{1}{6}, \quad B_3(x) = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x. \tag{6}$$

Using these polynomials, we have from the last expression of the first line of (2):

$$\int_M^N (x - [x] - \tfrac{1}{2}) f'(x)\,dx = \sum_{n=M}^{N-1} \int_0^1 B_1(t) f'(n+t)\,dt = \sum_{n=M}^{N-1} \left[\frac{B_2(t)}{2} f'(n+t)\Big|_0^1 - \frac{1}{2}\int_0^1 B_2(t) f''(n+t)\,dt \right], \tag{7}$$

where the relation (5) with $n = 2$ is used. By noting $B_2(0) = B_2(1) (= \frac{1}{6})$, the first term of RHS in the above can be written as

$$\sum_{n=M}^{N-1} \frac{B_2(t)}{2} f'(n+t)\Big|_0^1 = \frac{B_2(0)}{2} \sum_{n=M}^{N-1} [f'(n+1) - f'(n)] = \frac{B_2(0)}{2}\; f'(x)|_M^N. \tag{8}$$

The last term in (7) can be written, by defining the periodic function $B_2(x - [x]) = \overline{B}_2(x)$, as

$$\begin{aligned}\sum_{n=M}^{N-1} \frac{1}{2}\int_0^1 B_2(t) f''(n+t)\,dt &= \frac{1}{2}\int_M^N B_2(x-[x]) f''(x)\,dx = \frac{1}{2}\int_M^N \overline{B}_2(x) f''(x)\,dx = \frac{1}{2\cdot 3}\int_M^N \overline{B}_3'(x) f''(x)\,dx \\ &= \frac{1}{2\cdot 3}\,\overline{B}_3(x) f''(x)\big|_M^N - \frac{1}{2\cdot 3}\int_M^N \overline{B}_3(x) f'''(x)\,dx.\end{aligned} \tag{9}$$

The identity $B_3(0) = B_3(1) = 0$ implies $\overline{B}_3(n) = 0$ for any integer n, making the first term in RHS of (9) disappear. Thus, (7) can be simplified as

$$\int_M^N (x - [x] - \tfrac{1}{2}) f'(x)\,dx = \frac{B_2(0)}{2}\; f'(x)|_M^N . + \frac{1}{2\cdot 3}\int_M^N \overline{B}_3(x) f'''(x)\,dx. \tag{10}$$

By carrying integration by parts further, we obtain

$$\begin{aligned}\sum_{n=M}^{N} f(n) = \int_M^N f(x)\,dx &+ \frac{1}{2}[f(M) + f(N)] + \frac{B_2(0)}{2}\; f'(x)|_M^N - \frac{B_3(0)}{2\cdot 3}\; f''(x)|_M^N + \cdots + (-1)^k \frac{B_k(0)}{k!}\; f^{(k-1)}(x)\Big|_M^N \\ &+ (-1)^{k+1}\frac{1}{k!}\int_M^N \overline{B}_k(x) f^{(k)}(x)\,dx.\end{aligned} \tag{11}$$

The Bernoulli polynomial $B_n(x)$ reduces to the Bernoulli number B_n at $x = 0$. The Bernoulli numbers B_n are defined as the coefficient of the Taylor series expansion of the generating function $\frac{te^t}{e^t-1}$ (see e.g. [1] Chapter 1,[2], p. 11, [4], p.7)

$$\frac{te^t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}. \tag{12}$$

Another definition of the Bernoulli numbers is by the following recursion:

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = n, \quad n = 1, 2, 3, \ldots. \tag{13}$$

The first several B_n are

$$B_0 = 1, \quad B_1 = -\tfrac{1}{2}, \quad B_2 = \tfrac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}. \tag{14}$$

Furthermore, the Bernoulli numbers B_n with odd n are all zero, except for B_1, i.e.,

$$B_{2k+1} = 0 \ \text{ for all } \ k \geq 1. \tag{15}$$

Substituting these properties, we finally obtain what is known as the *Euler-Maclaurin summation formula*:

$$\sum_{n=M}^{N} f(n) = \int_M^N f(x)\,dx + \tfrac{1}{2}[f(M) + f(N)] + \frac{B_2}{2}\, f'(x)\big|_M^N + \frac{B_4}{4!}\, f'''(x)\big|_M^N + \cdots + \frac{B_{2\nu}}{(2\nu)!}\, f^{(2\nu-1)}(x)\Big|_M^N + R_{2\nu}, \tag{16}$$

where the error term can be expressed by setting $k = 2\nu$ in (11)

$$R_{2\nu} = -\frac{1}{(2\nu)!}\int_M^N \overline{B}_{2\nu}(x) f^{(2\nu)}(x)\,dx. \tag{17}$$

Since the term containing $B_{2\nu+1}$ can be ignored, we may alternatively set $k = 2\nu + 1$, yielding

$$R_{2\nu} = R_{2\nu+1} = \frac{1}{(2\nu+1)!}\int_M^N \overline{B}_{2\nu+1}(x) f^{(2\nu+1)}(x)\,dx. \tag{18}$$

A simple estimate of the error term $|R_{2\nu}|$ is obtained if $f^{(2\nu+1)}(x)$ is monotone decreasing, by using the fact that $\overline{B}_{2\nu+1}(x)$ alternates in sign as $\overline{B}_3(x)$ does. Edwards (cf. [2], p. 105) states "It is a general rule of thumb in applying the Euler-Maclaurin summation formula that as long as the terms are decreasing rapidly in size, the bulk of the errors is in the first term omitted."

2 Evaluation of $\zeta(s)$ by Euler-Maclaurin summation

We continue to follow the exposition by Edwards [2], Section 6.4. Instead of directly applying the Euler-Maclaurin summation method to $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, we consider summing $f(x) = x^{-s}$ from $x = N$ to infinity:

$$\begin{aligned}\sum_{n=N}^{\infty} n^{-s} = \zeta(s) - \sum_{n=1}^{N-1} n^{-s} &= \int_N^{\infty} f(x)\,dx + \tfrac{1}{2}[f(N) + f(\infty)] + \int_N^{\infty} \overline{B}_1(x) f'(x)\,dx \\ &= \int_N^{\infty} x^{-s}\,dx + \tfrac{1}{2}N^{-s} - s\int_N^{\infty} \overline{B}_1(x) x^{-s-1}\,dx \\ &= \frac{N^{1-s}}{s-1} + \tfrac{1}{2}N^{-s} + \frac{B_2}{2} sN^{-s-1} + \cdots + \frac{B_{2\nu}}{(2\nu)!} s(s+1)\cdots(s+2\nu-2)N^{-s-2\nu+1} + R_{2\nu},\end{aligned} \tag{19}$$

where

$$R_{2\nu} = \frac{1}{(2\nu+1)} \int_N^\infty \overline{B}_{2\nu+1}(x) f^{(2\nu+1)}(x)\,dx = -\frac{s(s+1)\cdots(s+2\nu)}{(2\nu+1)!} \int_N^\infty \overline{B}_{2\nu+1}(x) x^{-s-2\nu-1}\,dx. \quad (20)$$

By rearranging (19), we have the Euler-Maclaurin sum representation for $\zeta(s)$:

$$\zeta(s) = \sum_{n=1}^{N-1} n^{-s} + \frac{N^{1-s}}{s-1} + \tfrac{1}{2}N^{-s} + \frac{B_2}{2} sN^{-s-1} + \cdots + \frac{B_{2\nu}}{(2\nu)!} s(s+1)\cdots(s+2\nu-2)N^{-s-2\nu+1} + R_{2\nu}. \quad (21)$$

If N is about the same size as $|s|$, the terms of the series (21) decreases quite rapidly, so we expect the reminder $R_{2\nu}$ to be quite small. We wish to obtain an upper bound of its magnitude.

$$\begin{aligned} |R_{2\nu}| &= \left| \frac{s(s+1)\cdots(s+2\nu)}{(2\nu+1)!} \int_N^\infty \overline{B}_{2\nu-1}(x) x^{-s-2\nu+1}\,dx \right| \\ &= \left| \frac{s(s+1)\cdots(s+2\nu-1)}{(2\nu)!} \int_N^\infty [B_{2\nu+2} - \overline{B}_{2\nu+2}(x)] x^{-s-2\nu-2}\,dx \right| \\ &\le \frac{|s(s+1)\cdots(s+2\nu-1)|}{(2\nu)!} \int_N^\infty \left| B_{2\nu+2} - \overline{B}_{2\nu+2}(x) \right| x^{-\sigma-2\nu-2}\,dx. \end{aligned} \quad (22)$$

where the second line of the above was obtained by noting that $[B_{2\nu+2} - \overline{B}_{2\nu+2}(x)]' = -(2\nu+2)\overline{B}_{2\nu+1}(x)$, and $B_{2\nu+2} - \overline{B}_{2\nu+2}(N) = 0$ after integration by parts, and where $\sigma = \Re(s)$ in the last line.

It can be shown that $B_{2\nu+2} - B_{2\nu+2}(x)$ is zero at $x = 0$ and $x = 1$ and has its extremum at $x = \frac{1}{2}$ and this is the only extremum in $0 < x < 1$ (see e.g.,[2] p. 112). This implies that $B_{2\nu+2} - B_{2\nu+2}(x)$ never changes its sign, and has the same sign as $B_{2\nu+2}$ (because $B_{2\nu+2}(x)$ takes zero at $x = 0$ and $x = 1$). If we recall Euler's formula $\zeta(2\nu) = (-1)^{\nu-1}\dfrac{(2\pi)^{2\nu}}{2(2\nu)!}B_{2\nu}$ (see e.g. [4], p. 7, Eqn. (42)), we readily find that this sign is $(-1)^\nu$, i.e.,

$$|B_{2\nu+2} - \overline{B}_{2\nu+2}(x)| = (-1)^\nu [B_{2\nu+2} - \overline{B}_{2\nu+2}(x)]. \quad (23)$$

Thus, the integration in the last expression of (22) becomes

$$\begin{aligned} \int_N^\infty \left| B_{2\nu+2} - \overline{B}_{2\nu+2}(x) \right| x^{-\sigma-2\nu-2}\,dx &= \int_N^\infty (-1)^\nu [B_{2\nu+2} - \overline{B}_{2\nu+2}(x)] x^{-\sigma-2\nu-2}\,dx \\ &= |B_{2\nu+2}| \int_N^\infty x^{-\sigma-2\nu-2}\,dx - \int_N^\infty (-1)^\nu \overline{B}_{2\nu+2}(x) x^{-\sigma-2\nu-2}\,dx \\ &= \frac{|B_{2\nu+2}|}{\sigma+2\nu+1} N^{-\sigma-2\nu-1} - \frac{|\sigma+2\nu+2|}{2\nu+3} \int_N^\infty (-1)^\nu \overline{B}_{2\nu+3}(x) x^{-\sigma-2\nu-3}\,dx \\ &\le \frac{|B_{2\nu+2}|}{|\sigma+2\nu+1|} N^{-\sigma-2\nu-1}. \end{aligned} \quad (24)$$

By combining (22) and (24), we finally obtain

$$\begin{aligned} |R_{2\nu}| &\le \frac{\left| s(s+1)\cdots(s+2\nu+1) B_{2\nu+2} N^{-\sigma-2\nu-1} \right|}{(2\nu+2)!|\sigma+2\nu+1|} \\ &= \frac{|s+2\nu+1|}{|\sigma+2\nu+1|} \cdot \left| \frac{B_{2\nu+2}}{(2\nu+2)!} s(s+1)\cdots(s+2\nu) N^{-s-2\nu-1} \right|. \end{aligned} \quad (25)$$

Note that the $B_{2\nu+2}$ term is the first term omitted in the series representation (21). In other words, the error term $R_{2\nu} = R_{2\nu+1}$ is at most $\dfrac{|s+2\nu+1|}{|\sigma+2\nu+1|}$ times the magnitude of the first term omitted. Edwards ascribes this result to Backlund's dissertation in 1916 (see [2], p. 115).

Although the Euler-Maclaurin summation formula (21) was derived from the definition $\zeta(s) = \sum_{n=1}^\infty n^{-s}$ which is defined only for $\Re(s) > 1$, the formula is valid throughout the half-plane $\Re(s + 2\nu + 1) > 1$, in which the integral for $R_{2\nu}$ converges. Since ν is arbitrary, this provides an alternative proof for analytical continuation of $\zeta(s)$ throughout the entire s-plane, except at $s = 0$, which is a simple pole.

3 Evaluation of log differentials of $M(t) = |\zeta(\frac{1}{2} + it)|$

The materials we present in this section is new to the best of this author's knowledge. If we write

$$\zeta(\tfrac{1}{2} + it) = \sum_{n=1}^{\infty} n^{-\frac{1}{2}-it} = \sum_{n=1}^{\infty} \frac{\cos(t \log n)}{\sqrt{n}} - i \sum_{n=1}^{\infty} \frac{\sin(t \log n)}{\sqrt{n}} = A(t) - iB(t), \tag{26}$$

its modulus $M(t)$ is given by

$$M(t) = |\zeta(\tfrac{1}{2} + it)| = \sqrt{A^2(t) + B^2(t)}, \tag{27}$$

and its first and second differentials are

$$M'(t) = \frac{A(t)A'(t) + B(t)B'(t)}{M(t)}, \quad \text{and} \quad M''(t) = \frac{A(t)A''(t) + {A'}^2(t) + B(t)B''(t) + {B'}^2(t) - {M'}^2(t)}{M(t)}, \tag{28}$$

respectively. Thus, the log-differential of the modulus is

$$\frac{d \log M(t)}{dt} = \frac{M'(t)}{M(t)} = \frac{A(t)A'(t) + B(t)B'(t)}{A^2(t) + B^2(t)}. \tag{29}$$

and the second log-differential is

$$\frac{d^2 \log M(t)}{dt^2} = \frac{A(t)A''(t) + {A'}^2(t) + B(t)B''(t) + {B'}^2(t)}{A^2(t) + B^2(t)} - 2\left(\frac{M'(t)}{M(t)}\right)^2. \tag{30}$$

The last expression can be alternatively written as

$$\frac{d^2 \log M(t)}{dt^2} = \frac{M''(t)}{M(t)} - \left(\frac{M'(t)}{M(t)}\right)^2. \tag{31}$$

Recall Hardy' Z-function ([3]. See also [2] p. 119, and [7]) defined by

$$Z(t) = e^{i\vartheta(t)}\zeta(\tfrac{1}{2} + it), \tag{32}$$

where $\vartheta(t)$ is the Riemann-Siegel theta function:

$$\vartheta(t) = \Im\{\log \Gamma(\tfrac{1}{4} + \tfrac{it}{2})\} - \frac{t \log \pi}{2} = \arg\{\zeta(\tfrac{1}{2} + it)\}. \tag{33}$$

Then, the following identity readily follows from the definition (32):

$$|Z(t)| = M(t). \tag{34}$$

Thus, we have

$$\frac{M'(t)}{M(t)} = \frac{|Z(t)|'}{|Z(t)|} = \frac{Z'(t)}{Z(t)}, \tag{35}$$

where the last equality holds because $Z(t)$ is a real function.

By expanding the right-hand side (RHS) of (32) and setting it equal to LHS, we find

$$Z(t) = A(t)\cos\vartheta(t) + B(t)\sin\vartheta(t), \tag{36}$$

and

$$A(t)\sin\vartheta(t) - B(t)\cos\vartheta(t) = 0, \tag{37}$$

from which we have

$$B(t) = A(t)\tan\vartheta(t). \tag{38}$$

By substituting this into (36), we find

$$Z(t) = \frac{A(t)}{\cos\vartheta(t)} = \frac{B(t)}{\sin\vartheta(t)}. \tag{39}$$

Therefore, we have

$$A(t) = Z(t)\cos\vartheta(t), \quad \text{and} \quad B(t) = Z(t)\sin\vartheta(t), \tag{40}$$

which could have been directly derived from (32).

By differentiating (39), we have

$$Z'(t) = \frac{A'(t)}{\cos\vartheta(t)} + \frac{A(t)\vartheta'(t)\sin\vartheta(t)}{\cos^2\vartheta(t)} \tag{41}$$

$$= \frac{B'(t)}{\sin\vartheta(t)} - \frac{B(t)\vartheta'(t)\cos\vartheta(t)}{\sin^2\vartheta(t)}. \tag{42}$$

Thus, by taking the ratio of $Z'(t)$ to $Z(t)$ of (39), we have

$$\frac{Z'(t)}{Z(t)} = \frac{A'(t)}{A(t)} + \vartheta'(t)\tan\vartheta(t) \tag{43}$$

$$= \frac{B'(t)}{B(t)} - \vartheta'(t)\cot\vartheta(t). \tag{44}$$

By differentiating the above, we find

$$\frac{d}{dt}\left(\frac{Z'(t)}{Z(t)}\right) = \frac{d}{dt}\left(\frac{A'(t)}{A(t)}\right) + \frac{d[\vartheta'(t)\tan\vartheta(t)]}{dt}$$

$$= \frac{A''(t)}{A(t)} - \left(\frac{A'(t)}{A(t)}\right)^2 + \vartheta''(t)\tan\vartheta(t) + [\vartheta'(t)\sec\vartheta(t)]^2. \tag{45}$$

Alternatively we have

$$\frac{d}{dt}\left(\frac{Z'(t)}{Z(t)}\right) = \frac{d}{dt}\left(\frac{B'(t)}{B(t)}\right) - \frac{d[\vartheta'(t)\cot\vartheta(t)]}{dt}$$

$$= \frac{B''(t)}{B(t)} - \left(\frac{B'(t)}{B(t)}\right)^2 - \vartheta''(t)\cot\vartheta(t) + [\vartheta'(t)\csc\vartheta(t)]^2. \tag{46}$$

The function $\vartheta(t)$ can have the following Stirling series approximation (see Edwards [2] pp. 119-120):

$$\vartheta(t) = \frac{t}{2}\log\frac{t}{2\pi e} - \frac{\pi}{8} + \frac{1}{48t} + \frac{7}{5760t^3} + \cdots. \tag{47}$$

4 Evaluation of $\zeta'(s)$

The Euler-Maclaurin summation of $\zeta'(s)$ can be obtained in a manner similar to what we discussed in the preceding section. Since the $n = 1$ term disappear with differentiation, we have

$$\zeta'(s) = \sum_{n=2}^{\infty}\frac{d}{ds}n^{-s} = \sum_{n=2}^{\infty}\frac{d}{ds}e^{-s\log n} = -\sum_{n=2}^{\infty}(\log n)n^{-s}. \tag{48}$$

Thus, by defining $f(x)$ as

$$f(x) = -(\log x)x^{-s} = -(\log x)e^{-s\log x} = a(x)b(x), \quad \text{where} \quad a(x) = \log x, \quad \text{and} \quad b(x) = -x^{-s}. \tag{49}$$

its n-th derivative is given by

$$f^{(n)} = \sum_{k=0}^{n} \binom{n}{k} a^{(n-k)}(x) b^{(k)}(x), \tag{50}$$

where

$$a^{(j)}(x) = (-1)^{j-1}(j-1)!x^{-j}, \quad j \geq 1, \tag{51}$$

and

$$b^{(k)}(x) = s(s+1)(s+2)\cdots(s+k-1)(-1)^{k-1}x^{-s-k}, \quad k \geq 1. \tag{52}$$

Therefore, by substituting (51) and (52) into (50), we find

$$\begin{aligned} f^{(n)}(x) &= a^{(n)}(x)b(x) + \sum_{k=1}^{n-1} \binom{n}{k} a^{(n-k)}(x)b^{(k)}(x) + a(x)b^{(n)}(x) \\ &= (-1)^n(n-1)!x^{-n-s} + \sum_{k=1}^{n-1}(-1)^n \binom{n}{k}(n-k-1)!s(s+1)\cdots(s+k-1)x^{-n-s} \\ &\quad + (-1)^n s(s+1)\cdots(s+n-1)\log x \cdot x^{-s-n} \\ &= (-1)^n(n-1)!x^{-n-s}\left[1 - s(s+1)\cdots(s+n-1)\log x + \sum_{k=1}^{n-1}\frac{n!}{(n-k)k!}s(s+1)\cdots(s+k-1)\right], \quad n \geq 1. \end{aligned} \tag{53}$$

Thus,

$$f^{(n)}(x)\Big|_N^\infty = (-1)^{n-1}(n-1)!N^{-n-s}\left[1 - s(s+1)\cdots(s+n-1)\log N + \sum_{k=1}^{n-1}\frac{n!}{(n-k)k!}s(s+1)\cdots(s+k-1)\right], \quad n \geq 1. \tag{54}$$

By noting

$$\begin{aligned} \int_N^\infty f(x)\,dx &= -\int_N^\infty (\log x)x^{-s}\,dx = -\int_N^\infty \frac{(x^{-s+1})'}{s-1}\log x\,dx \\ &= \left[\frac{x^{-s+1}\log x}{s-1}\right]_N^\infty + \int_N^\infty \frac{x^{-s+1}}{s-1}\frac{1}{x}\,dx = \frac{N^{-s+1}\log N}{s-1} + \int_N^\infty \frac{x^{-s}}{s-1}\,dx \\ &= \frac{N^{-s+1}\log N}{s-1} + \frac{N^{-s+1}}{(s-1)^2} = \frac{N^{-s+1}}{s-1}\left[\log N + \frac{1}{s-1}\right], \end{aligned} \tag{55}$$

and

$$\tfrac{1}{2}[f(N) + f(\infty)] = -\tfrac{1}{2}N^{-s}\log N, \quad \text{for} \quad \sigma = \Re(s) > 0, \tag{56}$$

we have the following expression for the Euler-Maclaurin sum series for $\zeta'(s)$:

$$\begin{aligned} \zeta'(s) &= -\sum_{n=2}^{N-1}(\log n)n^{-s} + \frac{N^{-s+1}}{s-1}\left[\log N + \frac{1}{s-1}\right] - \tfrac{1}{2}N^{-s}\log N + \frac{B_2}{2}\, f'(x)|_N^\infty + \frac{B_4}{4!}\, f'''(x)|_N^\infty + \cdots \\ &\quad + \frac{B_{2\nu}}{(2\nu)!}\, f^{(2\nu-1)}(x)\Big|_N^\infty + R_{2\nu}, \end{aligned} \tag{57}$$

where $f^{(n)}(x)\big|_N^\infty$ '$(n \geq 1)$ is given by (54), and the error term is

$$R_{2\nu} - -\frac{1}{(2\nu)!}\int_N^\infty \overline{B}_{2\nu}(x)f^{(2\nu)}(x)\,dx. \tag{58}$$

References

[1] Arakawa, T., T. Ibukiyama, and M. Kaneko (2001), *Bernoulli Numbers and Zeta Function* (in Japanese), Makino Shoten, Tokyo.

[2] Edwards, H. M. (1974), *Riemann's Zeta Function*, Originally published by Academic Press in 1974. Republished by Dover Publications in 2001.

[3] Ivić, A. (2013), *Theory of Hardy's Z-Function*, Cambridge University Press.

[4] Kobayashi, H. (2015), "Towards a proof of the Riemann hypothesis, No. 1: Euler's zeta function," `http://hp.hisashikobayashi.com/#1`

[5] Kobayashi, H. (2016), "Towards a proof of the Riemann hypothesis, No. 5 Some results on the $\xi(s)$ and $\Xi(t)$ functions associated with Riemann's $\zeta(s)$ function," `http://hp.hisashikobayashi.com/#5` Also posted on arXiv: `http://arxiv.org/abs/1603.02954` (March 5, 2016)

[6] Kobayashi, H. (2016), "Towards a Proof of the Riemann Hypothesis, No. 6: Local Extrema of the $\Xi(t)$ Function and the Riemann Hypothesis." `http://hp.hisashikobayashi.com/#6` Also posted on arXiv: `http://arxiv.org/pdf/1603.02911.pdf` (March 5, 2016).

[7] Kobayashi, H. (2016), "Towards a Proof of the Riemann Hypothesis, No. 7: The $Z(t)$ function, Gram's law, Riemann-von Mangoldt formula, and Lehmer's phenomenon." `http://hp.hisashikobayashi.com/#7`

[8] Kobayashi, H. (2016), "Towards a Proof of the Riemann Hypothesis, No. 8: Negativity of $\frac{d^2}{dt^2}\log\Xi(t)$ and a conjecture on a sufficient condition for the Riemann hypothesis." `http://hp.hisashikobayashi.com/#8`

[9] MathWorks (2016), "Documentation: Riemann zeta function," `http://www.mathworks.com/help/symbolic/zeta.html`

[10] Odlyzko, A., "Tables of zeros of the Riemann zeta function," `http://www.dtc.umn.edu/~odlyZko/zeta_tables/index.html`.

[11] Riemann, B. (1859), "Über die Anzahl der Primizahlen under einer gegebener Grösse," *Motatsberichte. der Berliner Akademie*, November 1859, pp. 671-680. `http://www.maths.tcd.ie/pub/HistMath/People/Riemann/Zeta/EZeta.pdf`. An English translation "On the Number of Primes Less Than a Given Magnitude," can be also found in Appendix of Edwards [2], pp. 299-305.

[12] Titchmarsh, E.C. (1951), Revised by D. R. Heath-Brown (1986), *The Theory of the Riemann Zeta-function* (2nd Edition), Oxford University Press.

