No. 8: Negativity of $\frac{d^2}{dt^2} \log \Xi(t)$ and a conjecture on a sufficient condition for the Riemann hypothesis

Towards a Proof of the Riemann Hypothesis

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Abstract

In this report we show that if the Riemann hypothesis (RH) holds true, then the log-differential of $\Xi(t)$ is monotone decreasing in t between any two adjacent zeros of $\Xi(t)$, i.e., $\frac{d^2}{dt^2} \log \Xi(t) < 0$ for all t, where $\Xi(t) = \xi(\frac{1}{2} + it)$. This is an extension of our earlier results [5] that if RH holds, all local maximum of $\Xi(t)$ are positive and all local minima are negative.

Then after observing some numerical examples, we present a conjecture that $\frac{d^2}{dt^2} \log \Xi(t) < 0$ for entire t is also a sufficient condition for RH to hold.

Key words: $\Xi(t)$ function and its product representation, Monotone decreasing property of $\frac{d}{dt} \log \Xi(t)$, A necessary and sufficient conditions for the Riemann hypothesis.

1 Functions $\xi(s)$ and $\Xi(t)$ and the Riemann hypothesis

The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \text{for} \quad \Re(s) > 1, \tag{1}$$

which has been extended to the entire s-domain, using analytic continuation (See Riemann [9] and Edwards [1]). The function $\xi(s)$ defined by

$$\xi(s) = \frac{s(s-1)}{2} \pi^{-s/2} \Gamma(s/2) \zeta(s) \tag{2}$$

is reflective in the sense

$$\xi(1-s) = \xi(s). \tag{3}$$

Therefore, its value on the critical line $\Re(s) = \frac{1}{2}$ is a real and symmetric function of t, which we denote by $\Xi(t)$ (see [10], p. 16):

$$\Xi(t) = \xi\left(\tfrac{1}{2} + it\right), \quad \text{and} \quad \Xi(-t) = \Xi(t). \tag{4}$$

The Riemann hypothesis (RH) is equivalent to the assertion "The zeros of $\Xi(t)$ are all real," which is indeed the way Riemann stated in his celebrated article in 1859 (see Riemann's article in Appendix of [1], p. 301).

From Hadamard's product form representation (see e.g., [1]. p. 47 and [3]), we can write the $\xi(s)$ function as

$$\xi(s) = \frac{1}{2} \prod_{n \in \mathbf{Z}_{\pm}} \left(1 - \frac{s}{\rho_n}\right) \tag{5}$$

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where $\mathbf{Z}_\pm = \{\pm 1, \pm 2, \pm 3, \ldots\}$, and $\{\rho_n\}$ is the set of countably infinite number of zeta-zeros. We write

$$\rho_n = \beta_n + i\gamma_n, \tag{6}$$

where it is known that $0 < \beta_n < 1$ for all n. We enumerate the zeta-zeros, starting with the one closest to the origin. Thus, $\rho_1 = \frac{1}{2} + i14.134725\ldots, \rho_2 = \frac{1}{2} + i21.022040\ldots, \rho_3 = \frac{1}{2} + i25.016858\ldots, \ldots$. For any integer $k > 0$, we know that $\rho_{-k} = \overline{\rho}_k$, the complex conjugate of ρ_k, is also a zeta-zero. By defining $\mathbf{Z}_+ = \{1, 2, 3, \ldots\}$ and $\mathbf{Z}_- = \{-1, -2, -3, \ldots\}$, we can write

$$\mathbf{Z}_\pm = \mathbf{Z}_+ \cup \mathbf{Z}_-. \tag{7}$$

Let $\mathbf{R}$ be the set of zeta zeros whose real parts are equal to $\frac{1}{2}$, and let $\overline{\mathbf{R}}$ be the set of zeta zeros whose real parts are not $\frac{1}{2}$. Then we can rewrite (5) as

$$\xi(s) = \xi(0) \prod_{\rho_n \in \mathbf{R}} \left(1 - \frac{s}{\rho_n}\right) \prod_{\rho_m \in \overline{\mathbf{R}}} \left(1 - \frac{s}{\rho_m}\right). \tag{8}$$

Thus, the Riemann hypothesis is equivalent to claiming that $\overline{\mathbf{R}}$ is an empty set.

2 A necessary condition for the Riemann hypothesis

For any $\rho_n = \frac{1}{2} + i\gamma_n$ in the set $\mathbf{R}$, its complex conjugate $\rho_{-n} = \overline{\rho_n} = \frac{1}{2} - i\gamma_n$ is also in $\mathbf{R}$. Thus, by combining this pair we define

$$f_n(s) = \left(1 - \frac{s}{\rho_n}\right)\left(1 - \frac{s}{\overline{\rho_n}}\right), \tag{9}$$

and denote by $F_n(t)$ its value on the critical line $s = \frac{1}{2} + it$:

$$F_n(t) = f_n(\tfrac{1}{2} + it) = \frac{(\gamma_n - t)(\gamma_n + t)}{|\rho_n|^2}. \tag{10}$$

Since $f_n(s)$ satisfies the reflective property $f_n(1-s) = f_n(s)$, $F_n(t)$ is a real and symmetric function of t. Taking the log-differential of the above equation, we find

$$\frac{d}{dt}\log F_n(t) = \frac{1}{t - \gamma_n} + \frac{1}{t + \gamma_n} = \frac{2t}{t^2 - \gamma_n^2}, \quad \text{for} \quad t \neq \pm\gamma_n, \tag{11}$$

and

$$\frac{d^2}{dt^2}\log F_n(t) = -\frac{1}{(t - \gamma_n)^2} - \frac{1}{(t + \gamma_n)^2} = -\frac{2(t^2 + \gamma_n^2)}{(t^2 - \gamma_n^2)^2} < 0, \quad \text{for} \quad t \neq \pm\gamma_n. \tag{12}$$

Thus, the function $\frac{d}{dt}\log F_n(t)$ is a monotone decreasing function of t, except at $t = \pm\gamma_n$, where the function is discontinuous and jumps from $-\infty$ to $+\infty$.

From (8) we find the function $\Xi(t)$ defined by (4) can be written as

$$\Xi(t) = \xi(0) \prod_{\rho_n \in \mathbf{R}} F_n(t). \tag{13}$$

Hence,

$$\frac{d}{dt}\log \Xi(t) = \sum_{\rho_n \in \mathbf{R}} \frac{d}{dt}\log F_n(t) = \sum_{\rho_n \in \mathbf{R}} \frac{2t}{t^2 - \gamma_n^2}, \quad \text{for} \quad t \neq \pm\gamma_n, \tag{14}$$

and

$$\frac{d^2}{dt^2}\log \Xi(t) = -\sum_{\rho_n \in \mathbf{R}} \frac{2(t^2 + \gamma_n^2)}{(t^2 - \gamma_n^2)^2} < 0, \quad \text{for} \quad t \neq \pm\gamma_n. \tag{15}$$

The above result leads to the following theorem:

Theorem 2.1 (A necessary condition for the Riemann hypothesis)**.** *If the Riemann hypothesis (RH) is true,* $\frac{d}{dt}\log\Xi(t)$ *is a monotone decreasing function of* t *in the interval of any consecutive zeta zeros.*

Proof. A proof should be straightforward from the above discussion. By taking the logarithm of (5) and differentiating it with respect to s, we have

$$\frac{d}{ds}\log\xi(s) = \sum_{n\in\mathbf{Z}_\pm}\frac{1}{s-\rho_n}, \tag{16}$$

By setting $s=\frac{1}{2}+it$ in (16), we obtain

$$-i\frac{d}{dt}\log\Xi(t) = \sum_{n\in\mathbf{Z}_\pm}\frac{1}{\frac{1}{2}+it-\rho_n} = \sum_{n\in\mathbf{Z}_\pm}\frac{\frac{1}{2}-\beta_n-i(t-\gamma_n)}{|\frac{1}{2}+it-\rho_n|^2}, \tag{17}$$

where we used the relation $\frac{d}{dt}\Xi(t) = i\,\frac{d}{ds}\xi(s)\big|_{s=\frac{1}{2}+it}$. By taking the imaginary part of (17), we find

$$\frac{d}{dt}\log\Xi(t) = \sum_{n\in\mathbf{Z}_\pm}\frac{t-\gamma_n}{|\frac{1}{2}+it-\rho_n|^2}, \quad -\infty<t<\infty. \tag{18}$$

If the Riemann hypothesis holds, $|\frac{1}{2}+it-\rho_n| = |t-\gamma_n|$. Then, (18) is simplified to

$$\frac{d}{dt}\log\Xi(t) = \sum_{n\in\mathbf{Z}_\pm}\frac{1}{t-\gamma_n} = \sum_{n\in\mathbf{Z}_+}\left(\frac{1}{t-\gamma_n}+\frac{1}{t+\gamma_n}\right) = \sum_{n\in\mathbf{Z}_+}\frac{2t}{t^2-\gamma_n^2}, \tag{19}$$

which leads to

$$\frac{d^2}{dt^2}\log\Xi(t) = -\sum_{n\in\mathbf{Z}_+}\frac{2(t^2+\gamma_n^2)}{(t^2-\gamma_n^2)^2} < 0, \quad \text{for all } t. \tag{20}$$

□

The following corollary (i.e., Theorem 1.1 we proved in [5]) immediately follows from the above theorem.

Corollary 2.1 (Properties of local extrema of $\Xi(t)$ as a necessary condition for the Riemann hypothesis)**.** *If the Riemann hypothesis (RH) is true, local maxima of the function* $\Xi(t)$ *are all positive, and local minimum are all negative.*

Proof. If t^* is a local extremum, then $\Xi'(t^*)=0$, thus $\log\Xi(t^*)=0$. Using the identity

$$\frac{\Xi''(t)}{\Xi(t)} = \frac{d^2}{dt^2}\log\Xi(t) + \left(\frac{d}{dt}\log\Xi(t)\right)^2, \tag{21}$$

we readily find from (20)

$$\frac{\Xi''(t^*)}{\Xi(t^*)} = -\sum_{n\in\mathbf{Z}_+}\frac{2({t^*}^2+\gamma_n^2)}{({t^*}^2-\gamma_n^2)^2} < 0. \tag{22}$$

Thus, any local maximum must be positive and a local minimum must be negative. □

Example 1. In Figure 1 we show an illustrating example of $\Xi'(t)/\Xi(t)$ (in green) and $\frac{d}{dt}\left(\Xi'(t)/\Xi(t)\right)$ (in blue) for $t\in[279.0, 285.0]$, the same interval as in Report 7 [6], where we discussed Lehmer's phenomenon (see e.g., Edwards [1] pp. 176-177, and Ivić [2], pp. 26-28). In this interval the following four zeta zeros are known to exist on the critical line. They are the 126th through 129th zeros, enumerated from the origin. The imaginary components are:

$$\gamma_{126} = 279.2292\ldots, \quad \gamma_{127} = 282.4651\ldots, \quad \gamma_{128} = 283.2111\ldots \quad \gamma_{129} = 284.8359\ldots. \tag{23}$$

We computed $\Xi(t)$ by setting $s=\frac{1}{2}+it$ in (2), using the zeta function available in the Matlab Library [7]. Differentiation is approximated by a simple difference operation, using the increment of $\delta t = 0.005$. Note the similarity of these curves to those for Hardy's $Z(t)$ function reported in [6] Figure 6.

The green curve $\Xi'(t)/\Xi(t)$ is monotone decreasing within each interval between two successive zero points (γ_n, γ_{n+1}). The blue curve shows how the slope of the green curve varies. □

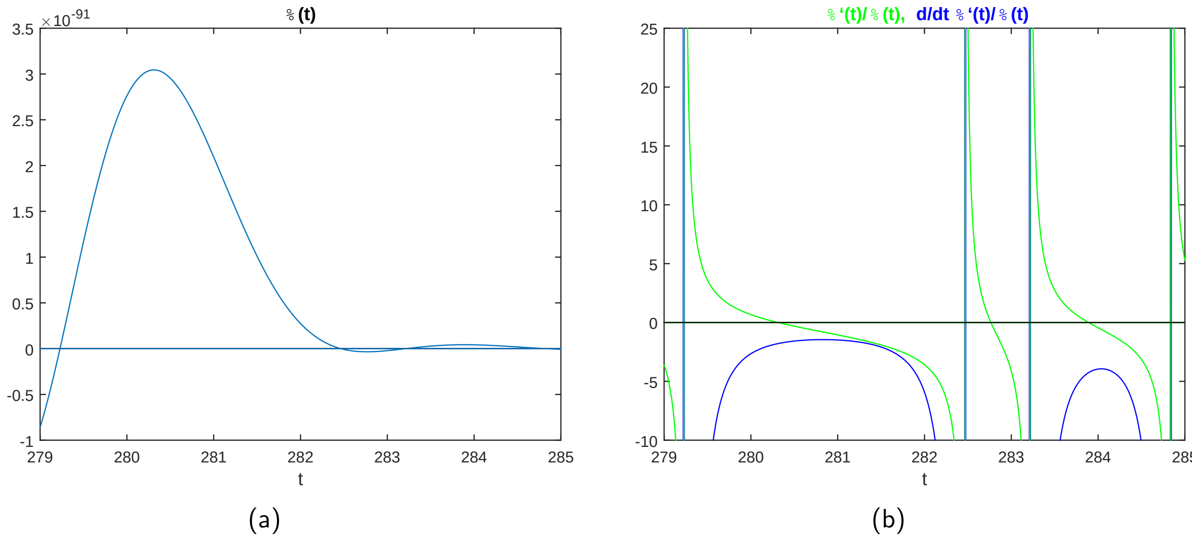


Figure 1: (a) $\Xi(t)$ for $t \in [279, 285]$; (b) $\frac{d}{dt}\log \Xi(t)$ (green) and $\frac{d^2}{dt^2}\log \Xi(t)$ (blue) for $t \in [279, 285]$.

3 If the Riemann hypothesis is false

For a zero $\rho_m = \beta_m + i\gamma_m$ in the set $\overline{\mathbf{R}}$, its reflective image around the critical line $\hat{\rho}_m = 1 - \beta_m + i\gamma_m$ and their complex conjugates $\overline{\rho_m} = \rho_{-m} = \beta_m - i\gamma_m$ and $\overline{\hat{\rho}_m} = \hat{\rho}_{-m} = 1 - \beta_m - i\gamma_m$ are also in the set $\overline{\mathbf{R}}$. Thus, by combining the four terms in the product-form representation (8), we define

$$\begin{aligned} g_m(s) &= \left(1 - \frac{s}{\rho_m}\right)\left(1 - \frac{s}{\hat{\rho}_m}\right) \cdot \left(1 - \frac{s}{\overline{\rho_m}}\right)\left(1 - \frac{s}{\overline{\hat{\rho}_m}}\right) \\ &= \frac{(s-\rho_m)(s-\hat{\rho}_m)\cdot(s-\overline{\rho_m})(s-\overline{\hat{\rho}_m})}{|\rho_m|^2|\hat{\rho}_m|^2}. \end{aligned} \tag{24}$$

It is not difficult to see that $g_m(s)$ is reflective, i.e., $g_m(1-s) = g_m(s)$. Then, by defining

$$G_m(t) = g_m(\tfrac{1}{2} + it), \tag{25}$$

we see the function $G_m(t)$ is, like $F_n(t)$ of (10), a real and symmetric function of t, and can be expressed as

$$G_m(t) = \frac{A_m(t)B_m(t)}{D_m} > 0, \tag{26}$$

where

$$A_m(t) = \left|\tfrac{1}{2} + it - \rho_m\right|^2 = (t-\gamma_m)^2 + (\beta_m - \tfrac{1}{2})^2, \tag{27}$$

$$B_m(t) = \left|\tfrac{1}{2} + it - \overline{\rho_m}\right|^2 = (t+\gamma_m)^2 + (\beta_m - \tfrac{1}{2})^2 = A_m(-t), \tag{28}$$

and $D_m = |\rho_m|^2|\hat{\rho}_m|^2$. From (26) it is clear that $G_m(t)$ is strictly positive for the entire $-\infty < t < \infty$.

The log-differential of (26) is given by:

$$\frac{d}{dt}\log G_m(t) = \frac{2(t-\gamma_m)}{A_m(t)} + \frac{2(t+\gamma_m)}{A_m(-t)} = \frac{4[t^2 - \gamma_m^2 + (\beta_m - \frac{1}{2})^2]}{A_m(t)A_m(-t)}. \tag{29}$$

By differentiating (29) once more, we find

$$\frac{d^2}{dt^2}\log G_m(t) = \frac{G_m''(t)}{G_m(t)} - \left(\frac{d}{dt}\log G_m(t)\right)^2 = \frac{4\left\{3t^2 - \gamma_m^2 + (\beta_m - \frac{1}{2})^2\right\}}{A_m(t)A_m(-t)} - \left(\frac{d}{dt}\log G_m(t)\right)^2 \tag{30}$$

The behavior of $G_m(t)$ is very different from that of $F_n(t)$. The log-differential $\frac{d}{dt}\log G_m(t) = G'_m(t)/G_m(t)$ is continuous for all t, but is neither monotone decreasing nor increasing. The point $t = \gamma_m$ is not a zero of $G_m(t)$ (recall $G_m(t) > 0$ for all t), but is very close to a minimum point t^* such that $G'_m(t^*) = 0$ when $\gamma_m \gg 1$:

$$t^* = \sqrt{\gamma_m^2 - (\beta_m - \tfrac{1}{2})^2} \approx \gamma_m - \frac{(\beta_m - \frac{1}{2})^2}{2\gamma_m} \approx \gamma_m, \quad \text{for } \ \gamma_m \gg 1. \tag{31}$$

By noting that the second term of (30) is zero at $t = t^*$, and that $3{t^*}^2 - \gamma_m^2 + (\beta_m - \frac{1}{2})^2 = 2\gamma_m^2 - 2(\beta_m - \frac{1}{2})^2 \approx 2\gamma_m^2$, $\ A_m(t^*) \approx (\beta_m - \frac{1}{2})^2$, and $A_m(-t^*) \approx 4\gamma_m^2$, we find at $t = t^*$

$$\frac{G''_m(t^*)}{G_m(t^*)} \approx \frac{2}{(\beta_m - \frac{1}{2})^2} > 0 \text{ for } \ \tfrac{1}{2} < \beta_m < 1, \tag{32}$$

The ratio (32) takes its infimum 8 as $\beta_m \to 1$. When $\beta_m \to \frac{1}{2}$ (i.e., when the zero ρ_m approaches the critical line), the ratio (32) diverge to $+\infty$, and the log-differential (29) and the second differential (30) reduce to

$$\lim_{\beta_m \to \frac{1}{2}} \frac{d}{dt}\log G_m(t) = \frac{4t}{t^2 - \gamma_m^2}, \tag{33}$$

and

$$\lim_{\beta_m \to \frac{1}{2}} \frac{d^2}{dt^2}\log G_m(t) = -\frac{4(t^2 + \gamma_m^2)}{(t^2 - \gamma_m^2)^2} < 0, \tag{34}$$

respectively. These results are not unexpected from (11) and (12), because as $\beta_m \to \frac{1}{2}$, the function $G_m(t) \to F_m^2(t)$ (recall $F_n(t)$ of (10)), because the pair of zeros ρ_m and $\hat{\rho}_m$ degenerate, in the limit $\beta_m \to \frac{1}{2}$, into a common zero of multiplicity two, lying on the critical line.

Example 2. Let us assume that the Riemann hypothesis should fail. Then there must exist at least one zeta zero of the form $\rho_m = \beta_m + i\gamma_m$, where $\frac{1}{2} < \beta_m < 1$. As an illustration purpose, let us consider the case where $\beta_m = 0.8$ and $\gamma_m = 281.0$. As remarked earlier, there should be a companion zero at the reflective position of ρ_m, i.e., $\hat{\rho}_m = 1 - \beta_m + i\gamma_m$. Furthermore, their complex conjugates $\overline{\rho_m}$ and $\overline{\hat{\rho}_m}$ should be also zeta zeros, which are located off the critical line.

Figure 2 (a) shows $G_m(t)$ (black), which becomes very close to zero at around $t = \gamma_m$, but remains strictly positive for the entire t. This property provides a crucial difference between $\frac{d}{dt}\log G_m(t)$ and $\frac{d}{dt}\log F_n(t)$; the latter holds under the validity of RH as discussed in the previous section. In Figure 2 (b), the log-differentials, $\frac{d}{dt}\log G_m(t)$ (green) and $\frac{d^2}{dt^2}\log G_m(t)$ (blue), are shown. It is quite clear that the green curve never diverges to $+\infty$ or $-\infty$, because $G_m(t)$ remains strictly positive. The blue curve, unlike its counterpart associated with $F_n(t)$, becomes positive around $t = \gamma_m$. Note that the minimum of $G_m(t)$ occurs at $t = t^* \approx \gamma_m$, and since both $G_m(t^*)$ and $G''_m(t^*)$ are positive, it follows that $G''_m(t^*)/G_m(t^*) > 0$. Thus, the blue curve has a positive peak at $t^* \approx \gamma_m$, and this peak value can be computed from (32) as approximately $2/(0.8 - 0.5)^2 = 22.22\ldots$ □.

We now consider a hypothetical situation where the ρ_m and its three associates are the only elements in the set $\overline{\mathbf{R}}$, which was defined in the text above (8). We construct a hypothetical $\xi_H(s)$[1] by

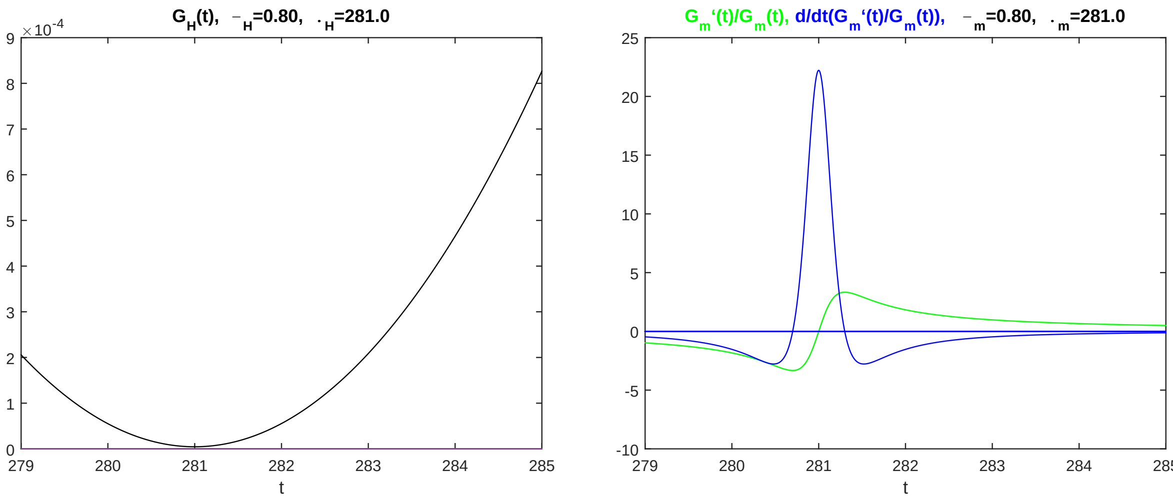
$$\xi_H(s) = \xi(s) g_m(s). \tag{35}$$

By defining $\Xi_H(t)$ as the value of $\xi_H(s)$ on the critical line, we find $\Xi_H(t)$ and its log-differentials are given by

$$\Xi_H(t) = \Xi(t) G_m(t), \tag{36}$$

$$\frac{d}{dt}\log \Xi_H(t) = \frac{d}{dt}\log \Xi(t) + \frac{d}{dt}\log G_m(t), \tag{37}$$

[1]One might argue that the same number of zeros in the set $\mathbf{R}$ should be removed in constructing $\xi_H(t)$. Such modification would not change the conclusion we will draw in the discussion below.



(a) (b)

Figure 2: (a) $G_m(t)$ for $t \in [279, 285]$, when $\beta_m = 0.8$ and $\gamma_m = 281.0$; (b) $\frac{d}{dt} \log G_m(t)$ (green) and $\frac{d^2}{dt^2} \log G_m(t)$ for $t \in [279, 285]$, with $\beta_m = 0.8$ and $\gamma_m = 281.0$.

and

$$\frac{d^2}{dt^2} \log \Xi_H(t) = \frac{d^2}{dt^2} \log \Xi_H(t) + \frac{d^2}{dt^2} \log G_m(t). \tag{38}$$

We expect that (37) is no longer monotone decreasing in t because of the non-monotone property of $\frac{d}{dt} \log G_m(t)$ of (29).

Example 3. Recall $\Xi(t)$ and its log differentials shown for $t \in [279, 285]$ in Example 1. We now insert the hypothetical zero ρ_m of Example 2. In Figure 3 (a) the function $\Xi_H(t)$ of (35) is shown. The log-differential functions $\frac{d}{dt} \log \Xi_H(t)$ and $\frac{d^2}{dt^2} \log \Xi_H(t)$ are shown in green and blue curves, respectively, in Figure 3(b). Clearly the function $\frac{d}{dt} \log \Xi_H(t)$ is not a monotone decreasing function. In Figure 3 (c) and (d) we show more details of (a) and (b) by plotting the curves in a narrower interval $[280.5, 281.5]$. We should note that t_H^* where a local minimum of $\Xi_H(t)$ occurs is slightly larger than $\gamma_m = 281.0$. This is because the value of $\frac{d}{dt} \log \Xi(t)$ is a small negative value (approximately -0.7 as seen in Figure 1 (b). When this negative value is superimposed to $\frac{d}{dt} \log G_H(t)$ of Figure 2(b), the zero-crossing point t_H^* of the green curve slightly move to the right as can be observed in Figure 3 (b) and (d).

4 Further discussion and a conjecture regarding a sufficient condition for RH

In Example 3, the location of the local exremum point of $G_m(t)$ is preserved, practically speaking, in $\Xi_H(t)$, i.e., $t_H^* \approx t_m^*$. Likewise, the property $\frac{G_m''(t_m^*)}{G_m(t_m^*)} > 0$ overrides $\frac{\Xi''(t)}{\Xi(t)} < 0$ in the vicinity of t_H^*, leading to $\Xi_H''(t_H^*)/\Xi_H(t_H^*) > 0$. This result is possible, because the log-differential operation tends to "localize" a function that takes the product-form such as (5). In other words, the behavior of $\frac{d}{dt} \log \Xi(t)$ in the interval of two adjacent zeros, say $t \in (\gamma_n, \gamma_{n+1})$, is largely unaffected by the behavior of $\Xi(t)$ outside of this interval (γ_n, γ_{n+1}).

The above observation leads us to state the following conjecture.

Conjecture 4.1. *(Conjecture regarding a sufficient condition for the Riemann hypothesis) If $\frac{d^2}{dt^2} \log \Xi(t) < 0$ for all t, then the Riemann hypothesis should hold true.*

The reason why we obtained the empirical observation in Example 3, which led us to the above conjecture, is because $\gamma_m = 281.0$ we chose is a mid-point between the zero on the left $\gamma_n = \gamma_{126} = 279.2292\ldots$ and the zero on the right $\gamma_{n+1} = \gamma_{127} = 282.4651\ldots$. If γ_m were very close to γ_n or γ_{n+1}, however, the positive peak of the blue curve in Example 2 might not be large enough, compared with the large negative value that the blue curve takes just above

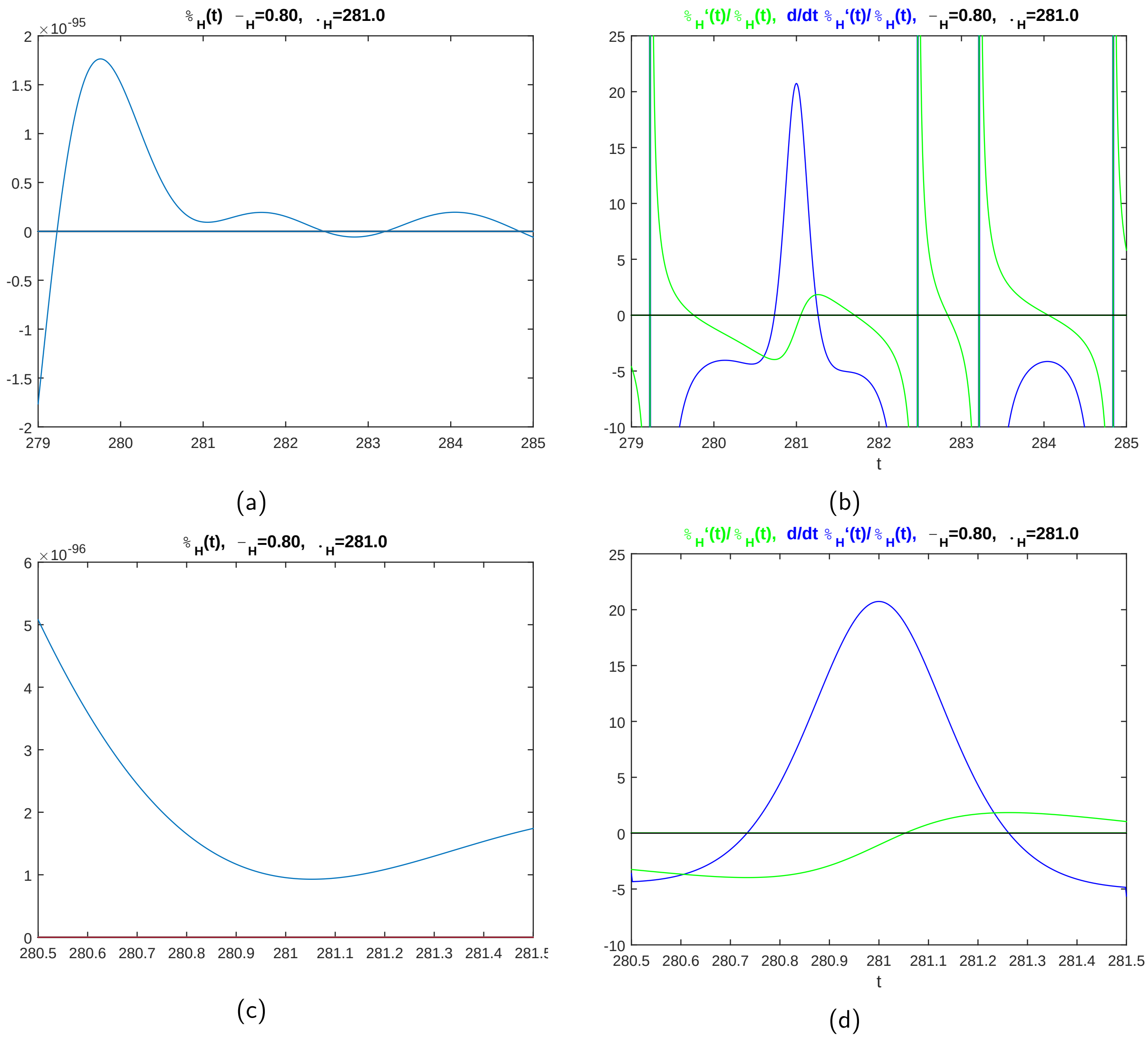


Figure 3: (a) $\Xi_H(t)$ for $t \in [279, 285]$, with a hypothetical zero $\rho_m = 0.8 + i281.0$; (b) the log-differentials of $\Xi_H(t)$ for $t \in [279, 285]$; (c) A zoomed-in version of (a) for $t \in [200.5, 201.5]$; (d) a zoomed-in version of (b) for $t \in [200.5, 201.5]$.

γ_n or below γ_{n+1}. What is required to prove the validity of the above conjecture is therefore to demonstrate that no zeta zero could exist so close to each other.

If we should be able to prove the above conjecture, then the remaining task to prove RH is to show $\frac{d^2}{dt^2} \log \Xi(t) < 0$ for all t. This seems to be a much more feasible than the approach we proposed earlier in [5], i.e., the one that relies on the properties of the local extrema of the function $\Xi(t)$.

References

[1] Edwards, H. M. (1974), *Riemann's Zeta Function*, Originally published by Academic Press in 1974. Republished by Dover Publications in 2001.

[2] Ivić, A. (2013), *Theory of Hardy's Z-Function*, Cambridge University Press.

[3] Kobayashi, H. (2016), "Towards a proof of the Riemann hypothesis, No. 3: Rieman's xi function and the product form representation," `http://hp.hisashikobayashi.com/#3`

[4] Kobayashi, H. (2016), "Towards a proof of the Riemann hypothesis, No. 5 Some results on the $\xi(s)$ and $\Xi(t)$ functions associated with Riemann's $\zeta(s)$ function," http://hp.hisashikobayashi.com/#5 Also posted on arXiv: http://arxiv.org/abs/1603.02954 (March 5, 2016)

[5] Kobayashi, H. (2016), "Towards a Proof of the Riemann Hypothesis, No. 6: Local Extrema of the $\Xi(t)$ Function and the Riemann Hypothesis." http://hp.hisashikobayashi.com/#6 Also posted on arXiv: http://arxiv.org/pdf/1603.02911.pdf (March 5, 2016).

[6] Kobayashi, H. (2016), "Towards a Proof of the Riemann Hypothesis, No. 7: The $Z(t)$ function, Gram's law, Riemann-von Mangoldt formula, and Lehmer's phenomenon." http://hp.hisashikobayashi.com/#7

[7] MathWorks (2016), "Documentation: Riemann zeta function," http://www.mathworks.com/help/symbolic/zeta.html

[8] Odlyzko, A., "Tables of zeros of the Riemann zeta function," http://www.dtc.umn.edu/~odlyZko/zeta_tables/index.html.

[9] Riemann, B. (1859), "Über die Anzahl der Primizahlen under einer gegebener Grösse," *Motatsberichte. der Berliner Akademie*, November 1859, pp. 671-680. http://www.maths.tcd.ie/pub/HistMath/People/Riemann/Zeta/EZeta.pdf. An English translation "On the Number of Primes Less Than a Given Magnitude," can be also found in Appendix of Edwards [1], pp. 299-305.

[10] Titchmarsh, E.C. (1951), Revised by D. R. Heath-Brown (1986), *The Theory of the Riemann Zeta-function* (2nd Edition), Oxford University Press.