No. 7: The $\mathbf{Z(t)}$ function, Gram's Law, Riemann-von Mangoldt Formula, and Lehmer's Phenomenon

Towards a Proof of the Riemann Hypothesis

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Abstract

In this report we will review various subjects related to $Z(t)$, known as Hardy's Z-function, closely related to the Riemann zeta-function. We also review empirical observation made by Gram in 1903 that the zeros of $Z(t)$ and the zeros of the Riemann-Siegel theta function $\sin\vartheta(t)$ interlace to each other.

We then show a derivation of the Riemann-von Manogoldt asymptotic formula for $N(T)$, the number of zeta zeros whose imaginary part is less than T. Our proof follows Backlund [1]. In the final section, we will discuss Lehmer's phenomenon, which shows that if the Riemann hypothesis is true, local maxima of $Z(t)$ cannot be negative and that its local minima cannot be positive.

Although we do not present any new theorems in this review report, several numerical examples and figures will be presented, which are designed to facilitate better understanding of the underlying theory.

In Appendix A, Jensen's theorem concerning the modulus of an analytic function averaged over a circle is discussed. This theorem is used in the proof of the Riemann-von Mangoldt formula. In Appendix B, we discuss Schwarz's lemma on the property of a holomorphic function on a disc; Borel-Carathéodori's theorem that shows the modulus of an analytic function over a disc is bounded by its real part and its value at the origin; Hadamard's three circles theorem, which states that the logarithm of the modulus of a holormorphic function on a circle of radius r is a convex function of $\log r$.

Key words: $Z(t)$ function, Riemann-Siegel theta function $\vartheta(t)$, Local extrema of $Z(t)$, Gram's points and Gram's law, Riemann-von Mangoldt formula, Backlund's estimate, Lehmer's phenomenon, Jensen's theorem, Schwarz's lemma, Borel-Carathéodori's theorem, and Hadamard's three circles theorem.

1 Introduction: The Functions $\Xi(t)$, $Z(t)$ and $\vartheta(t)$

The starting point of this review report is the result obtained in Section 4.3 of [9], which we recapitulate below. The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \text{for} \quad \Re(s) > 1, \tag{1}$$

which can then be extended to the entire s-domain, using analytic continuation (See Riemann [15] and Edwards [3]).

The function $\xi(s)$ is defined by

$$\xi(s) = g(s)\zeta(s), \tag{2}$$

where

$$g(s) = \frac{s(s-1)}{2}\pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2}). \tag{3}$$

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The function $\xi(s)$ is an entire function and possesses the following *reflective* (or *symmetric*) property:

$$\xi(1-s) = \xi(s), \tag{4}$$

which implies that $\xi(s)$ is real on the critical line, $\Re(s) = \frac{1}{2}$. Thus, by defining a real-valued function

$$\Xi(t) = \xi\left(\tfrac{1}{2} + it\right) = \Re\left\{\xi\left(\tfrac{1}{2} + it\right)\right\}, \tag{5}$$

we can state the Riemann hypothesis (RH), perhaps the most famous open problem in mathematics, as "The zeros of $\Xi(t)$ are all real," which is indeed the way Riemann stated his conjecture in his celebrated article in 1859 [15].

We write $g(s)$ of (3) as

$$g(s) = g_1(s)g_2(s), \quad \text{where} \quad g_1(s) = \tfrac{s(s-1)}{2} \quad \text{and} \quad g_2(s) = \pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2}). \tag{6}$$

Note that the function $g_1(s)$ is reflective, i.e.,

$$g_1(1-s) = g_1(s). \tag{7}$$

Therefore. its value on the critical line is real. Furthermore, it is a negative even function of t:

$$g_1(\tfrac{1}{2} + it) = g_1(\tfrac{1}{2} - it) = -\tfrac{1}{2}\left(t^2 + \tfrac{1}{4}\right) < 0. \tag{8}$$

Noting that it is the component $g_2(s)$ that transforms $\zeta(s)$ into the reflective function $\xi(s)$ (see (4)), we readily see that the function obtained by multiplying $g_2(s)$ to $\zeta(s)$ (see Riemann [15] p. 300, also see [9], Appendix A)

$$\eta(s) = g_2(s)\zeta(s) \tag{9}$$

also enjoys the same reflective property as $\xi(s)$ and $g_1(s)$:

$$\eta(1-s) = \eta(s). \tag{10}$$

Analogous to the $\Xi(t)$ function, we define a real-valued function

$$H(t) = \eta(\tfrac{1}{2} + it) = \Re\{\eta(\tfrac{1}{2} + it)\}, \tag{11}$$

which, similar to $\Xi(t)$, is an even function of t. By setting $s = \frac{1}{2} + it$ in (9) and (10), we readily find the following identity:

$$\frac{g_2(\frac{1}{2} + it)}{g_2(\frac{1}{2} - it)} = \frac{\zeta(\frac{1}{2} - it)}{\zeta(\frac{1}{2} + it)}. \tag{12}$$

Define a real-valued function

$$\vartheta(t) = \arg\{g_2(\tfrac{1}{2} + it)\} = \arg\left\{\Gamma\left(\tfrac{1}{4} + \tfrac{it}{2}\right)\right\} - \tfrac{t}{2}\log\pi, \tag{13}$$

which is known as the *Riemann-Siegel theta function*. We then define the function $Z(t)$ (e.g., [3], p. 119) by

$$Z(t) = \zeta\left(\tfrac{1}{2} + it\right)e^{i\vartheta(t)}. \tag{14}$$

Taking the absolute value of (14), we find

$$|Z(t)| = |\zeta(\tfrac{1}{2} + it)|. \tag{15}$$

Thus, $Z(t)$ has the same set of zeros on the real line $t \in \mathbb{R}$ as that of $\zeta(s)$ on the critical line $\Re(s) = \frac{1}{2}$. Thus, locating the Riemann zeros on the critical line is equivalent to locating zeros of $Z(t)$ on the real line $t \in \mathbb{R}$.

The $\zeta(s)$ function does not enjoy the reflective property like $\xi(s)$ or $\eta(s)$, thus the ratio

$$\chi(s) = \frac{\zeta(s)}{\zeta(1-s)} = \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\pi^{s-\frac{1}{2}}. \tag{16}$$

is not unity, but the following identity holds:

$$\chi(s)\chi(1-s) = 1, \quad \text{i.e.,} \quad \chi(1-s) = \chi(s)^{-1}. \tag{17}$$

From (2), (4), (9), (10) and (16), we find

$$\chi(s) = \frac{g(1-s)}{g(s)} = \frac{g_2(1-s)}{g_2(s)}. \tag{18}$$

The following identities are known for the gamma function (see e.g., [6], p. 3):

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}, \tag{19}$$

and

$$\Gamma(s)\Gamma(s+\tfrac{1}{2}) = 2^{1-2s}\sqrt{\pi}\;\Gamma(2s). \tag{20}$$

By setting $s \to \frac{s}{2}$ in (19), and $s \to \frac{1-s}{2}$ in (20), we obtain

$$\Gamma(\tfrac{s}{2})\Gamma(1-\tfrac{s}{2}) = \frac{\pi}{\sin(\frac{\pi s}{2})}, \tag{21}$$

$$\Gamma(\tfrac{1-s}{2})\Gamma(1-\tfrac{s}{2}) = 2^s\sqrt{\pi}\;\Gamma(1-s). \tag{22}$$

Taking the ratio of (22) to (21), we find

$$\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} = 2^s\pi^{-\frac{1}{2}}\Gamma(1-s)\sin(\tfrac{\pi s}{2}), \tag{23}$$

whose substitution into (16) leads to

$$\chi(s) = 2^s\pi^{s-1}\sin(\tfrac{\pi s}{2})\Gamma(1-s). \tag{24}$$

By using the formula $\sin(\pi s) = 2\sin(\frac{\pi s}{2})\cos(\frac{\pi s}{2})$ in (19) and substituting it into (24), we find another formula for $\chi(s)$:

$$\chi(s) = \frac{(2\pi)^s}{2\cos(\frac{\pi s}{2})\Gamma(s)}. \tag{25}$$

By setting $s = \frac{1}{2} + it$ in (18), we find the following relation between $\chi(s)$ and $\vartheta(t)$.

$$\chi(\tfrac{1}{2}+it) = \frac{g_2(\frac{1}{2}-it)}{g_2(\frac{1}{2}+it)} = e^{-2i\vartheta(t)}, \tag{26}$$

where we used the following results:

$$g_2\left(\tfrac{1}{2}+it\right) = |g_2(\tfrac{1}{2}+it)|e^{i\vartheta(t)}, \tag{27}$$

$$g_2\left(\tfrac{1}{2}-it\right) = |g_2(\tfrac{1}{2}-it)|e^{-i\vartheta(t)}, \tag{28}$$

and

$$|g_2(\tfrac{1}{2}+it)| = |g_2(\tfrac{1}{2}-it)| = \pi^{-\frac{1}{4}}|\Gamma(\tfrac{1}{4}+\tfrac{it}{2})|. \tag{29}$$

Note that the $\vartheta(t)$ function defined by (13) is an *odd function* of t, i.e.,

$$\vartheta(-t) = -\vartheta(t), \tag{30}$$

which can be shown from its definition (13) or by using the relation (26) and the identity (17).

Equation (26), together with (14), results in the following *second expression* for $Z(t)$ (see e.g., [6], pp. 2-3)

$$Z(t) = \zeta(\tfrac{1}{2}+it)\left[\chi(\tfrac{1}{2}+it)\right]^{-1/2}, \tag{31}$$

From (17), we find the *third expression* for $Z(t)$:

$$Z(t) = \zeta(\tfrac{1}{2}+it)\left[\chi(\tfrac{1}{2}-it)\right]^{1/2}. \tag{32}$$

If we define the complex function $z(s)$ by

$$z(s) = \zeta(s)\chi^{-1/2}(s) = \zeta(s)\chi^{1/2}(1-s) = [\zeta(s)\zeta(1-s)]^{1/2} = \zeta(s)\left(\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}\right)^{1/2}\pi^{\frac{1}{4}-\frac{s}{2}}, \tag{33}$$

the function $z(s)$ is also reflective. i.e.,

$$z(1-s) = z(s), \tag{34}$$

and we can write $Z(t)$ as the evaluation of $z(s)$ on the critical line:

$$Z(t) = z\left(\tfrac{1}{2}+it\right). \tag{35}$$

The last expression for $z(s)$ in (33) suggests

$$\begin{aligned} Z(t) &= \zeta(\tfrac{1}{2}+it)\left(\frac{\Gamma(\frac{1}{4}+\frac{it}{2})}{\Gamma(\frac{1}{4}-\frac{it}{2})}\right)^{1/2}\pi^{-it/2} = \zeta(\tfrac{1}{2}+it)\left(\frac{e^{i\arg\{\log\Gamma(\frac{1}{4}+\frac{it}{2})\}}}{e^{-i\arg\{\log\Gamma(\frac{1}{4}-\frac{it}{2})\}}}\right)^{1/2}\pi^{-it/2} \\ &= \zeta(\tfrac{1}{2}+it)e^{i\arg\{\log\Gamma(\frac{1}{4}+\frac{it}{2})\}}e^{-it\log\pi/2} = \zeta(\tfrac{1}{2}+it)e^{i\vartheta(t)}, \end{aligned} \tag{36}$$

which is $Z(t)$ defined earlier in (14).

We can show that $Z(t)$ is an *even function*, by using (26) and the identity (12), obtaining

$$\begin{aligned} Z(t) &= \zeta(\tfrac{1}{2}+it)\left[\frac{g_2(\frac{1}{2}+it)}{g_2(\frac{1}{2}-it)}\right]^{1/2} = \zeta(\tfrac{1}{2}+it)\left[\frac{\zeta(\frac{1}{2}-it)}{\zeta(\frac{1}{2}+it)}\right]^{1/2} \\ &= \left[\zeta(\tfrac{1}{2}+it)\zeta(\tfrac{1}{2}-it)\right]^{1/2}, \end{aligned} \tag{37}$$

which leads to

$$Z(-t) = Z(t), \tag{38}$$

and

$$|Z(t)| = |\zeta(\tfrac{1}{2}+it)|, \tag{39}$$

which is (15).

It will be instructive to note the following parallel relations between the Riemann-Siegel theta function $\vartheta(t)$ and the three functions $\Xi(t)$, $H(t)$ and $Z(t)$:

$$Z(t) = z(\tfrac{1}{2}+it) = \zeta(\tfrac{1}{2}+it)e^{i\vartheta(t)}, \tag{40}$$

$$H(t) = \eta(\tfrac{1}{2}+it) = \zeta(\tfrac{1}{2}+it)|g_2(\tfrac{1}{2}+it)|e^{i\vartheta(t)} = |g_2(\tfrac{1}{2}+it)|\cdot Z(t), \tag{41}$$

$$\Xi(t) = \xi(\tfrac{1}{2}+it) = \zeta(\tfrac{1}{2}+it)g_1(\tfrac{1}{2}+it)|g_2(\tfrac{1}{2}+it)|e^{i\vartheta(t)} = -|g(\tfrac{1}{2}+it)|\cdot Z(t), \tag{42}$$

where $g_1(\frac{1}{2}+it)$ defined by (8) is a strictly negative real function, and $|g_2(\frac{1}{2}+it)|$ of (29) is a strictly positive function. Thus, the sign of $Z(t)$ is the same as the sign of $H(t)$, but is opposite to the sign of $\Xi(t)$. As noted earlier, the set of infinitely many zeros of $Z(t)$ on the real line is exactly the same as the zeros of $\zeta(s)$ on the critical line. It goes without saying that the same observation applies to the zeros of $H(t)$ and $\Xi(t)$, as well.

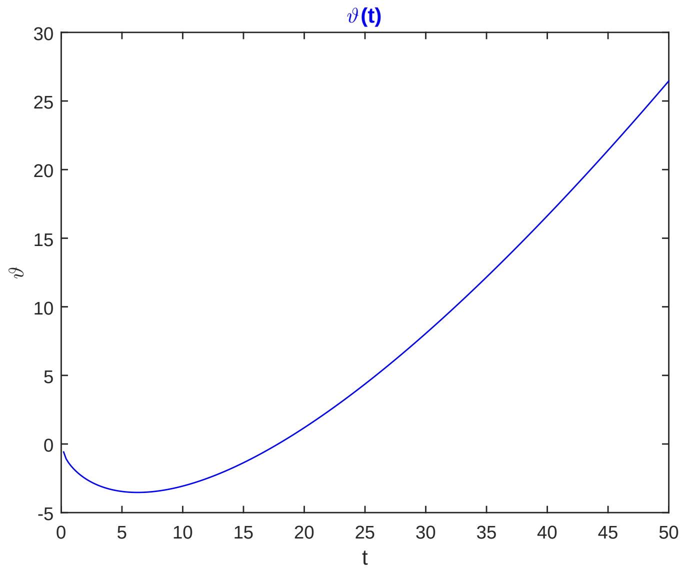


Figure 1: $\vartheta(t)$ for $0 \leq t \leq 50$.

2 Evaluation of the ζ zeros from $Z(t)$ and $\vartheta(t)$

The $\vartheta(t)$ function of (13) can be expanded using Stirling's series (see, Edwards [3], pp. 119-120, Ivic [6] p. 8):

$$\begin{aligned}\vartheta(t) &= \Im\{\log\Gamma(\frac{1}{4}+\frac{it}{2})\} - \frac{t\log\pi}{2}\\ &= \frac{t}{2}\log\frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + \frac{1}{48t} + \frac{7}{5760t^3} + O\left(t^{-5}\right)\end{aligned} \tag{43}$$

In Figure 1 we show a plot of the function $\vartheta(t)$ given by (43). The first two derivatives are

$$\vartheta'(t) = \frac{1}{2}\log\frac{t}{2\pi} + O(t^{-2}), \tag{44}$$

$$\vartheta''(t) = \frac{1}{2t} + O(t^{-3}). \tag{45}$$

Thus, $\vartheta(t)$ is monotone increasing for $t \geq 7 > 2\pi$.

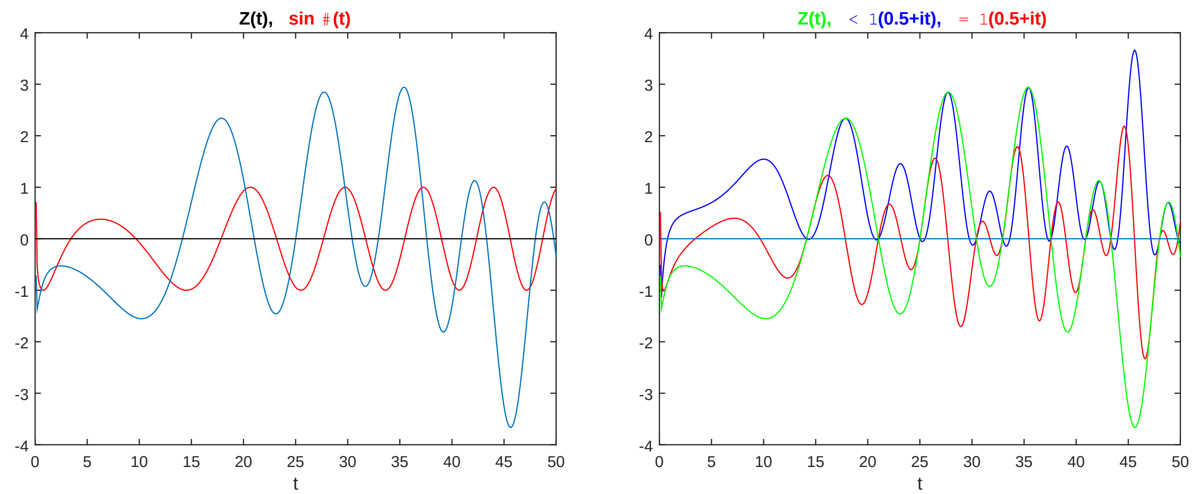
In order to locate the zeros of $\zeta(s)$ on the critical line, the following observations have been found useful in seeking efficient computational procedures:

1. By noting $n^{-\frac{1}{2}-it} = e^{-(\frac{1}{2}+it)\log n} = \frac{1}{\sqrt{n}}[\cos(t\log n) - i\sin(t\log n)]$, we can write

$$\Re\{\zeta(\tfrac{1}{2}+it)\} = 1 + \sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}\cos(t\log n). \tag{46}$$

   Note that the leading term 1 is dominant compared with the cosine functions whose amplitudes $n^{-1/2}$ slowly decay as n grows. Consequently, the real part of $\zeta(\frac{1}{2}+it)$ tends to have long positive intervals, interleaved with short negative intervals (sometimes zero intervals when $\Re\{\zeta(\frac{1}{2}+it)\}$ touches the zero line). See the blue curve in Figure 2 (b).

2. The imaginary part of $\zeta(\frac{1}{2}+it)$, on the other hand, consisting of infinitely many sine functions tends to oscillate between plus and minus evenly, and its zero crossings occur with sufficiently large slopes, and hence can be accurately estimated by interpolating the values of neighboring points. See the red curve in Figure 2 (b).



(a) (b)

Figure 2: (a) $Z(t)$ (blue) and $\sin\vartheta(t)$ (red) for $0 \le t \le 50$; (b) $Z(t)$ (green), $Re\{\zeta(0.5+it)\}$ (blue) and $\Im\{\zeta(0.5+it)\}$ (red).

3. From the relation

$$\Im\{\zeta(\tfrac{1}{2}+it)\} = -Z(t)\sin\vartheta(t), \tag{47}$$

we see that the zeros of $\Im\{\zeta(\frac{1}{2}+it)\}$ are composed of the zeros of $Z(t)$, denoted $\{t_n\}$, and the zeros of $\sin\vartheta(t)$, denoted $\{g_n\}$, i.e.,

$$\vartheta(g_n) = n\pi. \tag{48}$$

The points $\{g_n\}$ are called *Gram points* in deference to J. P. Gram [4] who observed in 1903 that $\{t_n\}$ and $\{g_n\}$ interleave to each other, except for the initial period where t is less than $t = 7$. In Figure 2 (a) we plot $Z(t)$ (in blue) and $\sin(\vartheta(t))$ (in red) for $0 \le t \le 50$. In view of Figure 1 and the definition of (48) we find label $g_{-1} = 9.6\ldots, g_0 = 17.8\ldots, g_1 = 23.2\ldots, \ldots, g_8 = 48.7\ldots$. Let us label the 10 zeros $Z(t)$ in this interval as $t_1(= 14.1\ldots), t_2(= 21.0\ldots), \ldots, t_9(= 48.0\ldots), t_{10}(= 49.7\ldots)$. Then by ignoring the smallest zero of $\sin(\vartheta(t)$, $t = 3.4\ldots$, which also corresponds to $n = -1$, we readily see $g_{-1} < t_1 < g_0 < t_2 < \ldots < t_9 < g_8 < t_{10}$. It was speculated by Gram that this alternating pattern of $\{t_n\}$ and $\{g_n\}$ would continue for some distance beyond $t = 50$. Hutchinson [5] (1925) who extended Gram's work and found the first 138 zeros of $Z(t)$. He referred to this alternative pattern as *Gram's law*.

"It is known today, however, that Gram's law fails infinitely often, What is correct is that on the average, there is exactly one zero of $Z(t)$ between two consecutive Gram points. Among the first billion and a half Gram intervals $(g_n, g_{n+1}]$, 72.6% have one zero, 13.8% have no zero, 13.4% have two zeros, 0.18% have three zeros, and that there are only 33 Gram intervals with four zeros..." (see Ivić [6], p. 112).

Furthermore, it has been empirically observed that

$$(-1)^n Z(g_n) = \Re\{\zeta(\tfrac{1}{2}+ig_n)\} > 0 \tag{49}$$

tends to hold, which is often referred to as the *weak Gram law*. In Figure 2(b), we plot $Re\{\zeta(0.5+it)\}(= Z(t)\cos\vartheta(t))$ (blue) and $\Im\{\zeta(0.5+it)\}(= -Z(t)\sin\vartheta(t))$ (red) as well as $Z(t)$ (green). Recall that $\zeta(0.5+it)$ and $Z(t)$ satisfy the simple relation (39).

4. Once we obtain approximate estimates of zeros of $\Im\{\zeta(\frac{1}{2}+it)\}$, we first check whether $\Re\{\zeta(\frac{1}{2}+it)\}$ changes its sign around these zeros. In the interval $0 \le t \le 50$, for instance, there are 21 zeros of

$\Im\{\zeta(\frac{1}{2}+it)\}$, ten out of which are also zeros of the real part $\Re\{\zeta(\frac{1}{2}+it)\}$. Hence, there are 10 zeta zeros on the critical line $s=\frac{1}{2}+it; 0\leq t\leq 50$. " To estimate these zeros more exactly, it suffices to estimate their position by linear interpolation and calculate $\Im\{\zeta(\frac{1}{2}+it)\}$ more precisely. For example, linear interpolation suggests, since $\Im\{\zeta(\frac{1}{2}+it)\}$ goes from -0.10 to $+0.05$ as t goes from 14.0 to 14.2, so that the zero lies two thirds of the way though the interval at 14.1333, and it is at this point that the value of $\Im\{\zeta(\frac{1}{2}+it)\}$ should be computed more exactly. This is precisely the method by which Gram computed the 15 roots given in Section 6.1..." (Edwards [3], p. 124).

"This program of Gram was followed by Hutchinson [5] who computed all the values g_n up to $g_{137}=300.468$ and determined the sign of $\Re\{\zeta(\frac{1}{2}+ig_n)\}(=Z(g_n)(-1)^n)$ for each of them. He found that there were two exceptions to the Gram weak law $\Re\{\zeta(\frac{1}{2}+ig_n)\}>0$, namely, $n=126$ ($g_{126}=282.455\ldots$) and $n=134$ ($g_{134}=295.584\ldots$).... He found, moreover, that these exceptions are slight in the sense that if the points are shifted only a little bit, then the sign of $\Re\zeta$ becomes positive. ..." ([3], pp. 126-127).

In Figure 3 (a) we plot $Z(t)$ (green), $\Re\{\zeta(\frac{1}{2}+it)\}$ (blue), and $\Im\{\zeta(\frac{1}{2}+it)\}$ (red) for the interval $280\leq t\leq 285$, and in the figure (b) its zoomed-in version $282.45\leq t\leq 282.47$. The left zero of the red curve is $g_{126}=282.455$, and the right zero is $t_{128}=282.465\ldots$. In the figure (b) the curve $\Re\{\zeta(\frac{1}{2}+it)\}$ (blue) completely overlaps with the curve $Z(t)$ (green), and hence is not visible.

Similarly, Figure 4 plot the curves for intervals that contain gram point $g_{134}=295.584\ldots$, which is the right zero in the figure (b). The left zero is $t_{135}=295.573\ldots$. Here again, the blue curve $\Re\{\zeta(\frac{1}{2}+it)\}$ is hidden behind the green curve $Z(t)$.

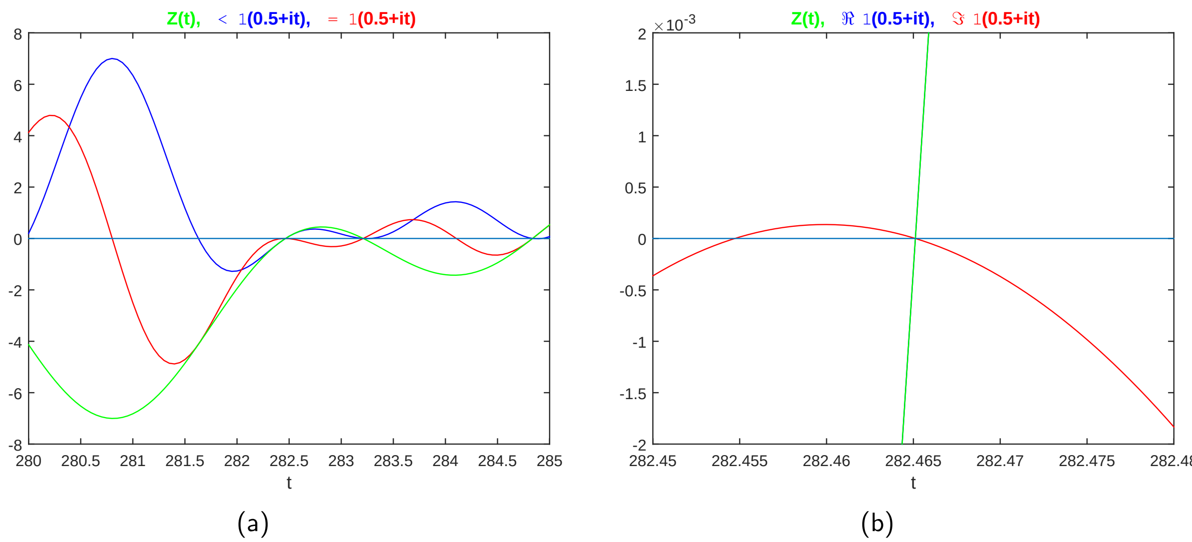


Figure 3: $Z(t)$ (green), $\Re\{\zeta(0.5+it)\}$ (blue), and $\Im\{\zeta(0.5+it)\}$ (red) for (a) $280\leq t\leq 285$; and (b) $282.45\leq t\leq 282.47$.

In a later report we shall discuss various techniques to compute the number of zeros and estimate their locations. Latest studies on Gram's law by Trudgian [17] (2011) and Korolev [10] (2011) are referred to by Ivić [6], p. 112, 120.

In the next section we will discuss another empirical phenomenon, known as Lehmer's phenomenon.

3 The Riemann-von Mangoldt Formula

In this section we discuss the distribution of the zeta-zeros in the range $0<t\leq T$ for any large T. Let $\mathcal{Z}$ be the set of zeros of the Riemann zeta-function $\zeta(s)$ that lie in the critical strip $\{s: 0<\Re(s)<1\}$. Note that $\overline{\zeta(s)}=\zeta(\overline{s})$. Because of this, and the reflective property of the Riemann zero, we see that if $\rho\in\mathcal{Z}$, then

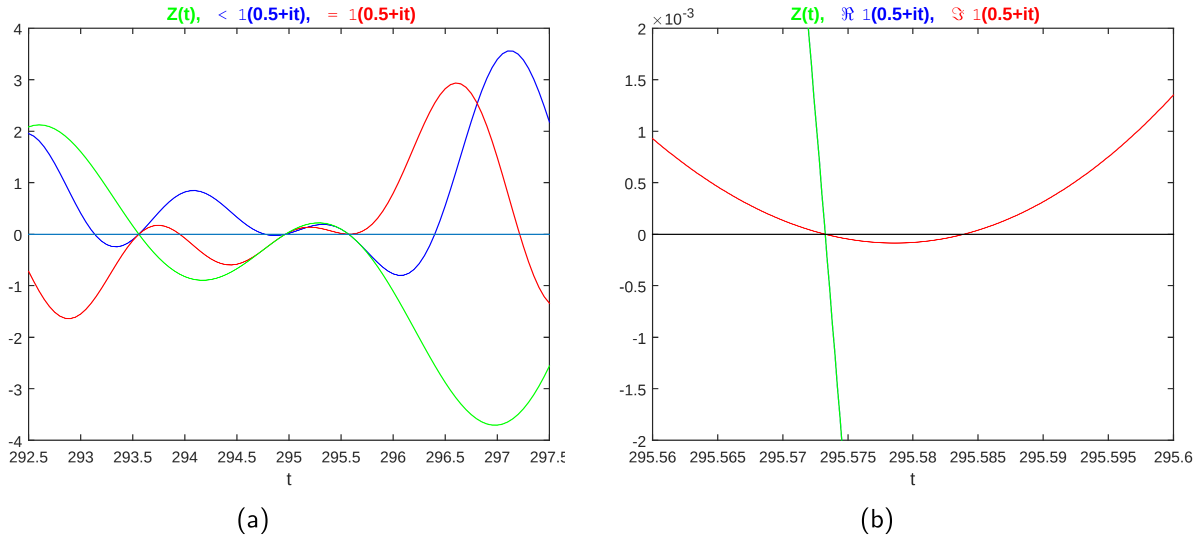


Figure 4: $Z(t)$ (green), $Re\{\zeta(0.5+it)\}$ (blue), and $\Im\{\zeta(0.5+it)\}$ (red) for (a) $292.5 \leq t \leq 297.5$; and (b) $295.56 \leq t \leq 295.60$.

$\overline{\rho}, 1-\rho, 1-\overline{\rho} \in \mathcal{Z}$. Note that we can write

$$\mathcal{Z} = \{\rho : \ \Im(\rho) > 0\} \cup \{1-\rho : \Im(\rho) > 0\} \tag{50}$$

Let $\mathcal{Z}_+$ be

$$\mathcal{Z}_+ = \{\rho \in \mathcal{Z} : \ \Im(\rho) > 0\}. \tag{51}$$

Let $N(T)$ be the number of zeta zeros whose imaginary part is between 0 and T, i.e.,

$$N(T) = |\{\rho \in \mathcal{Z}_+ : \ \Im(\rho) \leq T\}|. \tag{52}$$

Then, Riemann gave the following asymptotic formula in his 1859 paper [15], which was rigorously proved by von Mangoldt [12] in 1895:

Theorem 3.1 (Riemann-von Mangoldt)**.** *The number $N(T)$ of zeta-zeros ρ with $0 < \Im(\rho) \leq T$ is asymptotically given by*

$$N(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi e}\right) + \frac{7}{8} + O(\log T). \tag{53}$$

Proof. Recall the Cauchy's argument principle in complex analysis: if $f(s)$ is a meromorphic function inside and on some closed contour $\mathcal{C}$, and $f(s)$ has no zeros or poles on $\mathcal{C}$, then

$$\oint_{\mathcal{C}} \frac{f'(s)}{f(s)}\, ds = 2\pi i(N-P), \tag{54}$$

where N and P denote the number of zeros and poles, respectively, of $f(s)$ inside the contour $\mathcal{C}$.

Consider a rectangular region $\mathcal{R} = \{s = \sigma + it : -\epsilon \leq \sigma \leq 1+\epsilon, -T \leq t \leq T\}$, where ϵ is any positive number at this point. The function $\eta(s)$ of (9) can be written as $\eta(s) = \xi(s)/g_1(s)$, thus it has $2N(T)$ zeros (i.e.,$N(T)$ zeros with $\Im(\rho) > 0$ and the same number of zeros with $\Im(\rho) < 0$) and two poles (i.e., at $s = 0$ and $s = 1$) in the interior of $\mathcal{R}$. Then by applying the Cauchy's argument principle to $\eta(s)$, we have

$$\oint_{\partial\mathcal{R}} \frac{\eta'(s)}{\eta(s)}\, ds = 2\pi i(2N(T)-2) = 4\pi i(N(T)-1), \tag{55}$$

where $\partial\mathcal{R}$ is the contour which consists of the boundary of $\mathcal{R}$ oriented in the counterclockwise direction. Let us divide the contour $\partial\mathcal{R}$ into the following four L-shape paths:

$$\begin{aligned}
&\mathcal{L}_1: \ 1+\epsilon \to 1+\epsilon+iT \to \tfrac{1}{2}+iT \\
&\mathcal{L}_2: \ \tfrac{1}{2}+iT \to -\epsilon+iT \to -\epsilon \\
&\mathcal{L}_3: \ -\epsilon \to -\epsilon-iT \to \tfrac{1}{2}-iT \\
&\mathcal{L}_2: \ \tfrac{1}{2}-iT \to 1+\epsilon-iT \to 1+\epsilon
\end{aligned} \tag{56}$$

Recall that the function $\eta(s)$ is reflective (cf. (10)), Thus

$$\frac{\eta'(s)}{\eta(s)} = -\frac{\eta'(1-s)}{\eta(1-s)}. \tag{57}$$

Then, we find

$$\int_{\mathcal{L}_2} \frac{\eta'(s)}{\eta(s)}\,ds = -\int_{-\mathcal{L}_1} \frac{\eta'(1-s)}{\eta(1-s)}\,d(1-s) = \int_{\mathcal{L}_1} \frac{\eta'(s)}{\eta(s)}\,ds, \tag{58}$$

where $-\mathcal{L}_1$ is the reverse direction of $\mathcal{L}_1$. Similarly, we find

$$\int_{\mathcal{L}_3} \frac{\eta'(s)}{\eta(s)}\,ds = \int_{\mathcal{L}_4} \frac{\eta'(s)}{\eta(s)}\,ds. \tag{59}$$

Therefore, we have

$$\oint_{\partial\mathcal{R}} \frac{\eta'(s)}{\eta(s)}\,ds = 2\left(\int_{\mathcal{L}_1} + \int_{\mathcal{L}_4}\right) \frac{\eta'(s)}{\eta(s)}\,ds \tag{60}$$

The path $\mathcal{L}_4$ is equivalent to the reversed path of the complex conjugate of $\mathcal{L}_1$, i.e. $\mathcal{L}_4 = -\overline{\mathcal{L}_1}$. Thus, we find

$$\left(\int_{\mathcal{L}_1} + \int_{\mathcal{L}_4}\right) \frac{\eta'(s)}{\eta(s)}\,ds = \left(\int_{\mathcal{L}_1} - \int_{\overline{\mathcal{L}_1}}\right) \frac{\eta'(s)}{\eta(s)}\,ds = 2i\Im\left\{\int_{\mathcal{L}_1} \frac{\eta'(s)}{\eta(s)}\,ds\right\} \tag{61}$$

Then, from (60) and (55), we find

$$N(T) = 1 + \pi^{-1}\Im\left\{\int_{\mathcal{L}_1} \frac{\eta'(s)}{\eta(s)}\,ds\right\}. \tag{62}$$

Since we can write

$$\frac{\eta'(s)}{\eta(s)} = \frac{g_2'(s)}{g_2(s)} + \frac{\zeta'(s)}{\zeta(s)} = -\frac{\log\pi}{2} + \frac{\Gamma'\left(\frac{s}{2}\right)}{2\Gamma\left(\frac{s}{2}\right)} + \frac{\zeta'(s)}{\zeta(s)}, \tag{63}$$

(62) can be written as

$$\begin{aligned}
N(T) &= 1 + \pi^{-1}\Im\left\{\int_{\mathcal{L}_1}\left(-\frac{\log\pi}{2} + \frac{\Gamma'\left(\frac{s}{2}\right)}{2\Gamma\left(\frac{s}{2}\right)} + \frac{\zeta'(s)}{\zeta(s)}\right)ds\right\} \\
&= 1 - \frac{T}{2\pi}\log\pi + G(T) + S(T),
\end{aligned} \tag{64}$$

where the second term is obtained by noting

$$\Im\left\{\int_{\mathcal{L}_1} 1\,ds\right\} = \Im\{\tfrac{1}{2} + iT - (1+\epsilon)\} = T, \tag{65}$$

snd the last term $S(T)$ is

$$S(T) = \pi^{-1}\Im\left\{\int_{\mathcal{L}_1} \frac{\zeta'(s)}{\zeta(s)}\,ds\right\}, \tag{66}$$

which will be evaluated in the next subsection. The third term $G(T)$ of (64) can be readily evaluated as follows:

$$\begin{aligned} G(T) &= \pi^{-1}\Im\left\{\int_{\mathcal{L}_1} \frac{\Gamma'(\frac{s}{2})}{2\Gamma(\frac{s}{2})}\,ds\right\} = \pi^{-1}\Im\left\{\int_{\mathcal{L}_1} \frac{d}{ds}\log\Gamma(\tfrac{s}{2})\,ds\right\} = \pi^{-1}\Im\left\{\log\Gamma(\tfrac{s}{2})|_{s=\frac{1}{2}+iT} - \log\Gamma(\tfrac{s}{2})|_{s=1+\epsilon}\right\} \\ &= \pi^{-1}\Im\left\{\log\Gamma(\tfrac{1}{4}+\tfrac{iT}{2}) - \log\Gamma(\tfrac{1+\epsilon}{2})\right\} = \pi^{-1}\left(\vartheta(T) + \frac{T}{2}\log\pi\right), \end{aligned} \tag{67}$$

where we used (43) and the fact that $\Gamma(\frac{1+\epsilon}{2})$ is real. Thus,

$$\begin{aligned} N(T) &= 1 + \pi^{-1}\vartheta(T) + S(T) = 1 + \pi^{-1}\left(\frac{T}{2}\log\frac{T}{2\pi} - \frac{T}{2\pi}\right) - \frac{1}{8} + S(T) + O(T^{-1}) \\ &= \frac{T}{2\pi}\log\frac{T}{2\pi e} + \frac{7}{8} + S(T) + O(T^{-1}), \end{aligned} \tag{68}$$

where we used the Stirling approximation (43) applied to $\vartheta(T)$.

Thus, what remains to be done is to evaluate $S(T)$ of (66).

3.1 Evaluation of $S(T)$

We can write $S(T)$ as

$$\begin{aligned} S(T) &= \pi^{-1}\Im\left\{\int_{\mathcal{L}_1} \frac{d}{ds}\log\zeta(s)\,ds\right\} = \pi^{-1}\Im\left\{\int_{\mathcal{L}_1} \frac{d}{ds}[\log|\zeta(s)| + i\arg(\zeta(s))]\,ds\right\} \\ &= \pi^{-1}\int_{\mathcal{L}_1} \frac{d}{ds}\arg(\zeta(s))\,ds = \pi^{-1}\Delta_{\mathcal{L}_1}\arg(\zeta(s)), \end{aligned} \tag{69}$$

where $\Delta_{\mathcal{L}_1}\arg(\zeta(s))$ is the change in the $\arg(\zeta(s))$ along the path $\mathcal{L}_1$, starting from $s = 1+\epsilon$ and arriving at $s = \frac{1}{2} + iT$. We now split the path $\mathcal{L}_1$ into two straight lines: a vertial path $\mathcal{L}_V$ which starts from $s = 1+\epsilon$ and moves up to $s = 1+\epsilon+iT$; and a horizontal path $\mathcal{L}_H$, from $s = 1+\epsilon+iT$ towards the left until it reaches $s = \frac{1}{2} + iT$. So we can write (69) as

$$S(T) = \pi^{-1}[\Delta_{\mathcal{L}_V}\arg(\zeta(s)) + \Delta_{\mathcal{L}_H}\arg(\zeta(s))], \tag{70}$$

where

$$\Delta_{\mathcal{L}_V}\arg(\zeta(s)) = \int_{\mathcal{L}_V} \frac{d}{ds}\arg(\zeta(s))\,ds, \quad \text{and} \quad \Delta_{\mathcal{L}_H}\arg(\zeta(s)) = \int_{\mathcal{L}_H} \frac{d}{ds}\arg(\zeta(s))\,ds. \tag{71}$$

So far, the value of ϵ has not been specified. In his 1895 paper [12], von Mangoldt chose $\epsilon = 1$, which is also adopted by many authors (see e.g., Ivić [6], Matsumoto [14], Titchmarsh [16]). Here we follow Backlund (see e.g., Edwards [3], pp. 133-134), who chose $\epsilon = \frac{1}{2}$ in his 1914 paper [1]. Thus, $\mathcal{L}_V$ denotes the segment from $1\frac{1}{2}$ to $1\frac{1}{2} + iT$, and $\mathcal{L}_H$ denotes the segment from $1\frac{1}{2} + iT$ to $\frac{1}{2} + iT$. We know that $\zeta(s)$ does not have any zero on $\mathcal{L}_V$ since this path is outside the critical strip, into which all non-trivial zeta zeros are known to be confined.

If $\zeta(s)$ does not have any zero on the line segment $\mathcal{L}_V$ (where $\Im(s) = T$), then $t = T$ is not a discontinuity point of the step function $N(t)$. Under this assumption $\Delta_{\mathcal{L}_H}\arg(\zeta(s))$ lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.

If $\zeta(s)$ has k zeros on $\mathcal{L}_H$[1], the $\Delta_{\mathcal{L}_H}\arg(\zeta(s))$ lies between $-(k+\frac{1}{2})\pi$ and $(k+\frac{1}{2})\pi$. Thus,

$$\pi^{-1}|\Delta_{\mathcal{L}_H}\arg(\zeta(s))| < k + \tfrac{1}{2}. \tag{72}$$

[1] Note that the k zeros must line on the left-half segment of $\mathcal{L}_H$, i.e., $\rho_j \in [\frac{1}{2} + iT, 1 + iT)$; $j = 1, 2, \ldots, k$ since they must be within the critical strip, i.e., $0 < \Re(\rho_j) < 1$; $j = 1, 2, \ldots, k$.

We will show below that k, the number of zeros of $\Re\{\zeta(s)\}$ on the line segment $\mathcal{L}_H$ is at most $K \log T$ for $T \gg 1$ with some constant K.

Let us define

$$f(z) = \tfrac{1}{2}\left[\zeta(z+2+iT) + \zeta(z+2-iT)\right]. \tag{73}$$

For real z, $f(z) = \Re\{\zeta(z+2+iT)\}$. Thus, the number k in question is equal to the number of zeros of $f(z)$ on the interval $-1\frac{1}{2} \leq z \leq -1$[2]. Since $f(z)$ is analytic in the entire z-plane except for poles at $z = -1 \pm iT$ ($\zeta(z+2+iT)$ has $z = -1 - iT$ as a pole, and $\zeta(z+2-iT)$ has $z = -1 + iT$ as a pole, because $\zeta(1) = \infty$. Hence $f(-1 \pm iT) = \infty$). Thus, the following Jensen's formula (see Appendix A, (A.1) applies whenever $R \leq T$:

$$\log|f(0)| + \sum_{j=1}^{k} \log\left|\frac{R}{z_j}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(Re^{i\theta})\right|. \tag{74}$$

Let $R = 2 - \delta$, where δ should be chosen so that $f(z)$ has no zeros on the circle $|z| = R$. Since the zeros of $f(z)$ on the real axis are $-1\frac{1}{2} \leq z_j < -1;\quad j = 1, 2, \ldots, k$, $|R/z_j| \geq \dfrac{2-\delta}{3/2}$. Thus, we obtain from (74)

$$\log|f(0)| + k \log\frac{2(2-\delta)}{3} \leq \log M, \tag{75}$$

where M is the maximum value of $|f(z)|$ on $|z| = 2 - \delta$. As $\delta \to 0$, we have from (75) the following upper bound for k:

$$k \leq \frac{\log|M/f(0)|}{\log\frac{4}{3}}, \tag{76}$$

where M is an upper bound for $|f(z)|$ on $|z| = 2$. For $f(z)$ defined by (73), we find

$$|f(0)| = |\Re\{\zeta(2+iT)| = \left|1 + \sum_{n=2}^{\infty} \frac{\cos T \log n)}{n^2}\right| \geq 1 - \sum_{n=2}^{\infty}\frac{1}{n^2} \geq 2 - \zeta(2) = 2 - \frac{\pi^2}{6} > \frac{1}{4}, \quad \text{for all} \quad T \tag{77}$$

(As for Euler's result on $\zeta(2) = \pi^2/6$, see e.g.,(18) of [7]). This gives an upper bound on k:

$$k \leq \frac{\log 4 + \log M}{\log 4 - \log 3} = c_1 \log M + c_2, \quad \text{where} \quad c_1 = \frac{1}{\log 4 - \log 3}, \quad \text{and} \quad c_2 = \frac{\log 4}{\log 4 - \log 3}. \tag{78}$$

Now we wish to show that $\log M$ grows no faster than a constant times $\log T$ ([3]. p. 134). From the definition of M, we have

$$M = \tfrac{1}{2}\max\left|\left[\zeta(2+iT+2e^{i\theta}) + \zeta(2-iT+2e^{i\theta})\right]\right| \leq \max\left|\zeta(2+iT+2e^{i\theta})\right|. \tag{79}$$

In 1918 Backlund [2] obtained an estimate of the remainder R_2 in the Euler-Maclaurin summation for $\zeta(s)$, with $N = 1$ and $\nu = 0$ (see e.g., [3], p. 114):

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + R_2. \tag{80}$$

where

$$R_2 = -\frac{s(s+1)}{2}\int_1^{\infty} \overline{B}_2(x) x^{-s-2}\, dx, \tag{81}$$

[2]Although the line segment $\mathcal{L}_H$ is one unit long, i.e., from $1\frac{1}{2} + iT$ to $\frac{1}{2} + iT$, the zero ρ_j of $\zeta(s)$, if any, must be on the left half of this line segment ($\frac{1}{2} \leq \Re(\rho_j) < 1$, as stated in the preceding footnote. Thus, the corresponding zero on the real axis of z, denoted as $z_j = \Re(\rho_j) - 2$ must be $-1\frac{1}{2} \leq z_j < -1;\ j = 1, 2, \ldots, k$.

where $\overline{B}_2(x) = B_2(\{x\})$ is the periodified function of the Bernoulli polynomial $B_2(x) = x^2 - x + \frac{1}{6}$, and $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x. For the unit interval $0 \leq x < 1$, the Bernoulli polynomial $B_2(x)$ takes its maximum $B_2(0) = B_2 = \frac{1}{6}$ at $x = 0$ and its minimum $-\frac{1}{4} + \frac{1}{6} = -\frac{1}{12}$ at $x = \frac{1}{2}$, hence $|\overline{B}_2(x)| \leq B_2$ for $-\infty < x < \infty$. Thus, an upper bound for $|R_2|$ is

$$|R_2| \leq \left| \frac{B_2 s(s+1)}{2} \int_1^\infty x^{-s-2}\, dx \right| = \frac{|s(s+1)|}{12(\sigma+1)}, \tag{82}$$

where $\sigma = \Re(s) \geq \frac{1}{2}$ (because $s \in \mathcal{L}_H$), and $B_2 = B_2(0) = \frac{1}{6}$ is the Bernoulli constant (cf. e.g. [7], (44)). Thus, we find an upper bound for the absolute value of (80):

$$|\zeta(s)| \leq \frac{1}{|s-1|} + \frac{1}{2} + \frac{|s(s+1)|}{12(\sigma+1)}. \tag{83}$$

By setting $s = 2 + iT + 2e^{i\theta}$ in the above and using (79), we find

$$\begin{aligned} M &\leq \frac{1}{|1 + iT + 2e^{i\theta}|} + \frac{1}{2} + \frac{|2 + iT + 2e^{i\theta}||3 + iT + 2e^{i\theta}|}{12(1+\sigma)} \\ &\leq \frac{1}{T-2} + \frac{1}{2} + \frac{(T+2+4)(T+2+5)}{12(\frac{1}{2}+1)} \leq \frac{T^2}{18}(1 + O(T^{-1})). \end{aligned} \tag{84}$$

Therefor, we have

$$M = O(T^2), \quad \text{i.e.,} \quad \log M = O(\log T). \tag{85}$$

Thus, we have proved Theorem 3.1.

By refining the above estimate, Backlund [2] obtained the following result (see [3], p. 134):

$$\left| N(T) - \left(\frac{T}{2\pi} \log \frac{T}{2\pi e} + \frac{7}{8} \right) \right| < 0.137 \log T + 0.443 \log(\log T) + 4.350, \quad \text{for all} \quad T \geq 2. \tag{86}$$

□

4 Lehmer's Phenomenon.

In this section we will discuss what is known as Lehmer's phenomenon, which is concerned with local extrema of the function $Z(t)$. As can be seen in Figure 2a and (b), there is a local maximum of a negative value (≈ -0.526) at $t \approx 2.47\ldots$. We begin with the following proposition (see Edwards ([3] pp. 176-177) and Ivić ([6], pp. 26-28):

Theorem 4.1. *If the Riemann hypothesis (RH) is true, $\frac{Z'(t)}{Z(t)}$ is monotone decreasing between successive zeros of $Z(t)$.*

Proof. Equation (42) can be written as

$$\Xi(t) = -f(t)Z(t), \tag{87}$$

where

$$f(t) = |g(\tfrac{1}{2} + it)| = \tfrac{1}{2}\pi^{-\frac{1}{4}}(t^2 + \tfrac{1}{4})|\Gamma(\tfrac{1}{4} + \tfrac{it}{2})|. \tag{88}$$

By differentiating (87) and dividing it by (87), we find

$$\frac{\Xi'(t)}{\Xi(t)} = \frac{f'(t)}{f(t)} + \frac{Z'(t)}{Z(t)}, \tag{89}$$

from which we readily have

$$\frac{d}{dt}\frac{Z'(t)}{Z(t)}=\frac{d}{dt}\frac{\Xi'(t)}{\Xi(t)}-\frac{d}{dt}\frac{f'(t)}{f(t)}. \tag{90}$$

In our earlier report [9] on the product-form representation of $\xi(s)$, we obtained Eq. (25) of that report, which we reproduce here:

$$\frac{\xi'(s)}{\xi(s)}=\sum_{\rho\in\mathcal{Z}}\frac{1}{s-\rho}, \tag{91}$$

where ρ's are the zero of $\xi(s)$. If the RH is true, then $\Re(\rho)=\frac{1}{2}$ for all ρ's. By noting

$$\Xi'(t)=\frac{d}{dt}\xi(\tfrac{1}{2}+it)=i\xi'(s)|_{s=\frac{1}{2}+it},$$

and evaluating (91) at $s=\frac{1}{2}+it$, we find, by writing $\rho=\frac{1}{2}+i\gamma$

$$\frac{\Xi'(t)}{\Xi(t)}=\sum_{\gamma}\frac{1}{t-\gamma}, \quad \text{for } t\neq\gamma. \tag{92}$$

By differentiating the above equation once more, we find

$$\frac{d}{dt}\frac{\Xi'(t)}{\Xi(t)}=-\sum_{\gamma}\frac{1}{(t-\gamma)^2}<0, \quad \text{for } t\neq\gamma \tag{93}$$

In order to evaluate the second term of (90), we take the logarithm of (88) and differentiate it:

$$\begin{aligned}\frac{f'(t)}{f(t)}&=\frac{2t}{t^2+\frac{1}{4}}+\frac{d}{dt}\log|\Gamma(\tfrac{1}{4}+\tfrac{it}{2})|\\&=\frac{2t}{t^2+\frac{1}{4}}+\frac{d}{dt}\log\Re\left\{\Gamma(\tfrac{1}{4}+\tfrac{it}{2})\right\}\\&=\frac{2t}{t^2+\frac{1}{4}}+\Re\left\{\frac{i}{2}\left\{\frac{d}{ds}\log\Gamma(\tfrac{s}{2})\Big|_{s=\frac{1}{2}+it}\right\}\right\},\end{aligned} \tag{94}$$

where we used the following identity to derive the second line in the above:

$$\log|c(t)|=\Re\{\log c(t)\}. \tag{95}$$

Recall the Stirling formula for the gamma function (see [6], p. 27):

$$\log\Gamma(s)=(s-\tfrac{1}{2})\log s-s+\tfrac{1}{2}\log 2\pi+\sum_{j=1}^{K}\frac{(-1)^jB_{j+1}}{j(j+1)s^j}+O\left(\frac{1}{|s|^{K+1}}\right), \tag{96}$$

where K is any fixed integer $K\geq 1$, and the point $s=0$ and neighborhoods of the poles of $\Gamma(s)$ are excluded. The B_j are the Bernoulli numbers: $B_2=\frac{1}{6}, B_4=-\frac{1}{30}$, etc. Note also that $B_{2n+1}=0$ for all $n\geq 1$. Since the terms that involve B_2, B_4, etc. are much smaller than the leading terms, we have

$$\frac{d}{ds}\log\Gamma(\tfrac{s}{2})\approx\frac{1}{2}\log\frac{s}{2}-\frac{1}{2s}. \tag{97}$$

Thus,

$$\begin{aligned}\frac{d}{ds}\log\Gamma(\tfrac{s}{2})\Big|_{s=\frac{1}{2}+it}&\approx\frac{1}{2}[\log(\tfrac{1}{2}+it)-\log 2]-\frac{1}{1+2it}\\&=\frac{1}{2}\left[\log\sqrt{t^2+\tfrac{1}{4}}\left(\frac{\frac{1}{2}}{\sqrt{t^2+\frac{1}{4}}}+\frac{it}{\sqrt{t^2+\frac{1}{4}}}\right)-\log 2\right]-\frac{1-2it}{1+4t^2}.\end{aligned} \tag{98}$$

From this and (94), we find

$$\frac{d}{dt}\Re\{\log\Gamma(\tfrac{1}{4}+\tfrac{1}{2t})\} = \Re\left\{\frac{i}{2}\,\frac{d}{ds}\log\Gamma(\tfrac{s}{2})\Big|_{s=\frac{1}{2}+it}\right\} \approx -\frac{1}{8}\log(t^2+\tfrac{1}{4})\frac{t}{\sqrt{t^2+\frac{1}{4}}} - \frac{t}{4t^2+1}. \tag{99}$$

Thus, (94) becomes

$$\frac{f'(t)}{f(t)} \approx \frac{2t}{t^2+\frac{1}{4}} - \frac{1}{8}\log(t^2+\tfrac{1}{4})\frac{t}{\sqrt{t^2+\frac{1}{4}}} - \frac{t}{4t^2+1} = \frac{7t}{4t^2+1} - \frac{1}{8}\log(t^2+\tfrac{1}{4}) \approx \frac{7}{4t} - \frac{\log t}{4} \tag{100}$$

The second term dominates for $t > 4.6$, thus the function $\frac{f'(t)}{f(t)}$ is negative for $t > 4.6$. Thus,

$$-\frac{f'(t)}{f(t)} \approx \frac{\log t}{4}, \quad \text{for} \quad t \gg 1. \tag{101}$$

In Figure 5, we show an illustrative example of the curves $\frac{\Xi'(t)}{\Xi(t)}$ (green), $-\frac{f'(t)}{f(t)}$ (red), and $\frac{Z'(t)}{Z(t)}$ (blue) for the interval $279 \leq t \leq 285$. The function $-\frac{f'(t)}{f(t)} \approx \frac{\log t}{4}$ is a monotonically increasing function, but is nearly flat ($\approx 1.42\ldots$) in this interval. Thus, $\frac{Z'(t)}{Z(t)}$ is an upward shifted version of $\frac{\Xi'(t)}{\Xi(t)}$ by $-\frac{f'(t)}{f(t)} \approx 1.42\ldots$. It is apparent that the function $\frac{Z'(t)}{Z(t)}$ is monotone decreasing in any interval between two consecutive zeros

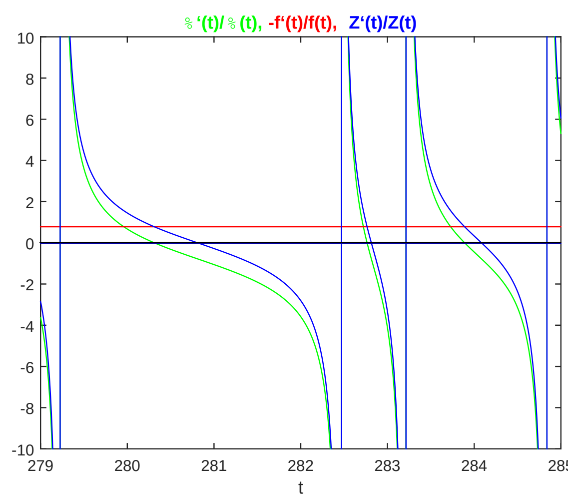


Figure 5: An example of $\frac{\Xi'(t)}{\Xi(t)}$ (green), $-\frac{f'(t)}{f(t)}$ (red) and $\frac{Z'(t)}{Z(t)}$ (blue): $279 \leq t \leq 285$.

t_n and t_{n+1} of $Z(t)$. In order to prove it mathematically, we need to show $\frac{d}{dt}\frac{Z'(t)}{Z(t)} < 0$. From (93), we know the first term of (90) is negative:

$$\frac{d}{dt}\frac{Z'(t)}{Z(t)} = -\sum_n \frac{1}{(t-t_n)^2} < 0. \tag{102}$$

Since it is known that $t_{n+1} - t_n \ll \frac{1}{\log\log t_n}$, it follows that

$$-\sum_n \frac{1}{(t-t_n)^2} < -C(\log\log t)^2,$$

where $C > 0$ (see e.g., Ivić [6] p. 27). The other term

$$-\frac{d}{dt}\frac{f'(t)}{f(t)} \approx \frac{d}{dt}\frac{\log t}{4} = \frac{1}{4t} \tag{103}$$

is strictly positive but is nearly zero. Thus,

$$\frac{d}{dt}\frac{Z'(t)}{Z(t)} \approx \frac{d}{dt}\frac{\Xi'(t)}{\Xi(t)} = -\sum_n \frac{1}{(t-t_n)^2} < 0. \tag{104}$$

In Figure 6 we show $\frac{d}{dt}\frac{Z'(t)}{Z(t)}$ (blue) for the same interval as Figure 5. $\frac{d}{dt}\frac{f'(t)}{f(t)} \approx \frac{1}{4t}$ is $8.8 - 9.0 \times 10^{-4}$ in this interval and virtually zero compared with $\frac{d}{dt}\frac{Z'(t)}{Z(t)} \approx \frac{\Xi'(t)}{\Xi(t)}$. □

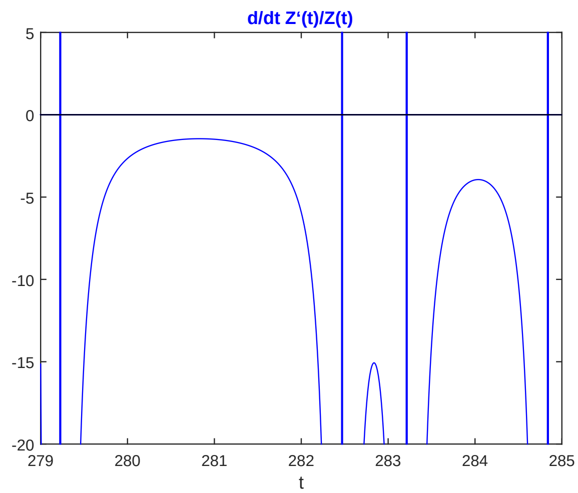


Figure 6: $\frac{d}{dt}\frac{Z'(t)}{Z(t)}$ (blue) in the interval $279 \leq t \leq 285$. Note that $\frac{\Xi'(t)}{\Xi(t)}$ ((green) cannot be seen, since it is indistinguishable from $\frac{d}{dt}\frac{Z'(t)}{Z(t)}$

The above theorem leads to the following proposition concerning the local extrema of $Z(t)$:

Corollary 4.1. *If the Riemann hypothesis is true, then $Z(t)$ cannot have a positive local minimum or a negative local maximum.*

Proof. We will proof this corollary by the method of contradiction. Suppose that $Z(t)$ has a positive local minimum or a negative local maximum between two consecutive zeros t_n and t_{n+1}. Then $Z'(t)$ must have at least two zeros, say u_1 and u_2 with $u_1 < u_2$ in (t_n, t_{n+1}). The above theorem implies, however, that

$$\frac{Z'(u_1)}{Z(u_1)} > \frac{Z'(u_2)}{Z(u_2)}. \tag{105}$$

But from the definition of u_1 and u_2, we have

$$\frac{Z'(u_1)}{Z(u_1)} = \frac{Z'(u_2)}{Z(u_2)} = 0, \tag{106}$$

which contradicts with the inequality (105). Thus we have proven the property of $Z(t)$ stated in this corollary. □

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Appendix A: Jensen's Theorem

In complex analysis. Jensen's theorem (also called Jensen's formula [3]) introduced by the Danish mathematician Johan Jensen (1859-1925) in 1899, relates the average modulus of an analytic function on a circle to the number of its zeros inside the circle.

Theorem A.1 (Jensen's Theorem)**.** *Let $f(z)$ be an analytic function in a disc of radius R centeed at the origin. Let $\rho_1, \rho_2, \ldots, \rho_n$ be the zeros (repeated according to multiplicity) of $f(z)$ inside the disc, and assume $f(0) \neq 0$. Then*

$$\log|f(0)| + \sum_{k=1}^{n} \log\left(\frac{R}{|\rho_k|}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta, \tag{A.1}$$

or equivalently

$$\log\left|f(0)\cdot\frac{R}{\rho_1}\cdot\frac{R}{\rho_2}\cdots\frac{R}{\rho_n}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta, \tag{A.2}$$

Another equivalent form of Jensen's formula is

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta - \log|f(0)| = \int_0^R \frac{n_f(0;r)}{r}\, dr, \tag{A.3}$$

where $n_f(z_0; r)$ is the number of zeros of $f(z)$ in the disc $z = r$ centered at $z = z_0$.

Proof. We follow Edwards ([3], p. 40). If $f(z)$ has no zeros inside the disc, i.e., if $n = 0$, then (A.2) reduces to

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{1\pi} \log|f(Re^{i\theta})|\, d\theta, \tag{A.4}$$

which states that the value of the $\log|f(z)|$ at the center of a disc is equal to the average value on the boundary circle. The formula (A.4) can be proved from Cauchy's integral formula:

$$\log f(0) = \frac{1}{2\pi i}\oint_{|z|=R} \frac{\log f(z)}{z}\, dz = \frac{1}{2\pi}\int_0^{2\pi} \log f(Re^{i\theta})\, d\theta, \tag{A.5}$$

where $\log f(z)$ is defined in the disc as

$$\log f(z) = \log f(0) + \int_0^z \frac{f'(s)}{f(s)}\, ds. \tag{A.6}$$

By observing that $\log|f(z)|$ is the real part of the analytic function $\log f(z)$, taking the real part of the Cauchy formula (A.5) leads to (A.4).

We now replace $f(z)$ in the mean formula (A.4) by $F(z)$ defined by

$$F(z) = f(z)\frac{R^2 - \overline{\rho}_1 z}{(z-\rho_1)R}\cdot\frac{R^2 - \overline{\rho}_2 z}{(z-\rho_2)R}\cdots\frac{R^2 - \overline{\rho}_n z}{(z-\rho_n)R}. \tag{A.7}$$

we have

$$\log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|F(Re^{i\theta})|\, d\theta. \tag{A.8}$$

By noting

$$\left|\frac{R^2 - \overline{\rho}_k z}{(z-\rho_k)R}\right| = \begin{cases} \left|\frac{R}{\rho_k}\right| & \text{for } z = 0 \\ 1 & \text{for } |z| = R. \end{cases} \tag{A.9}$$

[3]This should not be confused with Jensen's inequality often used in the theory of convex function and probability theory

we obtain Jensen's formula (A.2).

On the bounadary of the disc, (A.6) can take the form

$$\log f(Re^{i\theta}) = \log f(0) + \int_0^R \frac{df(re^{i\theta})}{f(re^{i\theta})}\, dr. \tag{A.10}$$

Taking the real part of the both sides and taking the average over θ, we have

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta = \log|f(0)| + \frac{1}{2\pi}\int_0^{2\pi} \Re\left\{\int_0^R \frac{df(re^{i\theta})}{f(re^{i\theta})}\, dr\right\} d\theta \tag{A.11}$$

By writing $df(re^{i\theta}) = f'(re^{i\theta})e^{i\theta}\, dr$ and $s = re^{i\theta}$, hence $d\theta = \dfrac{ds}{ire^{i\theta}}$, the second term of the RHS of the above expression becomes

$$\frac{1}{2\pi}\int_0^{2\pi} \Re\left\{\int_0^R \frac{df(re^{i\theta})}{f(re^{i\theta})}\, dr\right\} d\theta = \frac{1}{2\pi i}\int_0^R \frac{1}{r}\Re\left\{\oint_{|s|=r} \frac{f'(s)}{f(s)}\, ds\right\} dr. \tag{A.12}$$

Then by applying the Cauchy argument principle (54) to the circle $C:\ |s| = r$,

$$\frac{1}{2\pi i}\oint_{|s|=r} \frac{f'(s)}{f(s)}\, ds = n_f(0, r), \tag{A.13}$$

where $n_f(z_0; r)$ is the number of zeros of $f(z)$ found in the disc of radius r centered at $z = z_0$ as defined in (A.3). By substituting this result into (A.11), we obtain (A.3). □

Appendix B: Schwarz Lemma, Borel-Carathéodory Theoem and Hadamard Three-Circle Theorem

In this appendices we discuss two theorems that were used in the conventional evaluation of $S(T)$ (e.g., [6, 14, 16] (In our treatment discussed in Section 3.1 based on Backlund [1], however, these theorems are not used). Before we introduce the first one due to Émile Borel (1871-1956) and Consatantin Carathéodory (1873-1950), we need the following lemma by the German mathematician Hermann Schwarz (1843-1921).

Lemma B.1 (Schwarz)**.** *Let $\mathcal{D}_1$ be an open unit disc entered at the origin, i.e., $\mathcal{D}_1 = \{s : |s| < 1\}$, and let $f(s)$ be a holomorphic function which maps $\mathcal{D}_1$ to itself, with $f(0) = 0$. Then*

$$|f(s)| \le |s| \ \ \textit{for all}\ \ s \in \mathcal{D}_1, \ \ \text{and} \ \ |f'(0)| \le 1. \tag{B.1}$$

Moreover, suppose that $|f(s^*)| = |s^*|$ for some $s^* \in \mathcal{D}_1 \setminus \{0\}$[4]*, or if $|f'(0)| = 1$. Then, $f(s) = \alpha s$ for some constant α such that $|\alpha| = 1$.*

Proof. Let us define the function $g(s)$ by

$$g(s) = \begin{cases} \dfrac{f(s)}{s} & \text{for } \ s \ne 0, \\ f'(0) & \text{for } \ s = 0. \end{cases} \tag{B.2}$$

Then, $g(s)$ is differentiable at the origin, hence $g(s)$ is also holomorphic on $\mathcal{D}_1$. Let $\mathcal{C}_r = \{s : |s| \le r\}$ be a *closed* disc of radius r centered at the origin. Then the *maximum modulus principle*[5] implies that for $r < 1$ there exists s_r on the bounday of $\mathcal{C}_r$ such that

$$|g(s)| \le |g(s_r)| = \frac{|f(s_r)|}{|s_r|} \le \frac{1}{r}, \ \ \text{for all} \ \ s \in \mathcal{C}_r. \tag{B.3}$$

[4] The symbol \ is "setminus", thus $s^* \in \mathcal{D}_1$ but $s^* \ne 0$.

[5] If $f(s)$ is a holomorphic function, then the modulus $|f(s)|$ can have its maximum value only on the boundary of its domain $\mathcal{D}$, unless $f(s)$ is a constant function. In other words, for any point s_0 inside $\mathcal{D}$, there exist other points arbitrarily close to s_0 at which $|f(s)|$ takes larger values. The same holds for its minimum value.

The last inequality is due to the fact $f(s) \in \mathcal{D}_1$, and hence $|f(s)| \leq 1$ for all $s \in \mathcal{C}_r \subset \mathcal{D}_1$. As $r \to 1$, we have $|g(s)| \leq 1$, i.e.,

$$|f(s)| \leq |s| \quad \text{for all} \quad s \in \mathcal{D}_1 \setminus \{0\}, \quad \text{and } |g(0)| = |f'(0)| \leq 1. \tag{B.4}$$

Moreover, suppose $|f(s^*)| = |s^*|$ for some $s^* \in \mathcal{D}_1 \setminus \{0\}$, or if $|f'(0)| = 1$. Then the modulus $|g(s)|$ takes its maximum value, unity, at $s = s^*$ or $s = 0$, neither of which is on the boundary of $\mathcal{D}_1$. Then from the maximum modulus principle, $g(s)$ must be equal to some constant α such that $|\alpha| = 1$. Therefore, $f(s) = \alpha s$, as claimed. □

We are now in a position to introduce the Borel-Carathéodory theorem, which shows that an analytic function may be bounded by its real part. It is an application of the above Schwarz lemma and the maximum modulus principle. Émile Borel (1871-1956) was a French mathematician, and Consatantin Carathéodory (1873-1950) was a Greek mathematician who spent most of his life in Germany.

Theorem B.1 (Borel-Carathéodory). *Let $f(s)$ be a holomorphic function on $\mathcal{D}_R = \{s : |s| \leq R\}$, the disc of radius R, centered at the origin. Then for $0 < r < R$, the following inequality holds:*

$$\max_{|s|=r} |f(s)| \leq \frac{2r}{R-r} \sup_{|s| \leq R} \Re\{f(s)\} + \frac{R+r}{R-r}|f(0)|. \tag{B.5}$$

Proof. Let

$$A = \sup_{|s| \leq R} \Re\{f(s)\} \tag{B.6}$$

1. The case $f(0) = 0$:
   Since $\Re\{f(s)\}$ is a harmonic function, it must satisfy the maximum principle, thst is, it takes its maximum on the boundary $|s| = R$. Thus, $f(0) = 0$ cannot be the maximum, hence we have $A > 0$. The definition of A implies that $f(s)$ maps the entire complex plane $\mathcal{C}$ to the half-plane $P = \{s : \Re(s) \leq A\}$.

   Consider the following two mappings:

$$g_1(s) = \frac{s}{A} - 1, \quad \text{and} \quad g_2(s) = \frac{R(s+1)}{s-1}. \tag{B.7}$$

   The function g_1 maps P into the left half-plane $\mathcal{C}_- = \{s \in \mathcal{C}, \Re(s) \leq 0\}$, and g_2 maps $\mathcal{C}_-$ into the disc $\mathcal{D}_R$. The composite function of g_1 and g_2 defined by

$$g_3(s) = g_2(g_1(s)) = \frac{R\left(\frac{s}{A} - 1 + 1\right)}{\left(\frac{s}{A} - 1\right) - 1} = \frac{Rs}{s - 2A} \tag{B.8}$$

   maps P into $\mathcal{D}_R$ with $g_3(0) = 0$. We then consider the composite h of f and g_3:

$$h(s) = g_3(f(s)) = \frac{Rf(s)}{f(s) - 2A}, \tag{B.9}$$

   which is a holomorphic function, which has $\mathcal{D}_R$ as both its domain and range. Since we are assuming $f(0)$ in this part of the proof $h(0) = 0$. Then by applying Schwarz's lemma to $h(s)$, we find

$$\frac{|Rf(s)|}{|f(s) - 2A|} \leq |s|, \tag{B.10}$$

   For $|s| \leq r$, the above becomes

$$R|f(s)| \leq r|f(s) - 2A| \leq r|f(s)| + 2Ar, \tag{B.11}$$

   from which we find

$$|f(s)| \leq \frac{2Ar}{R-r}, \tag{B.12}$$

   which is the special case of (B.5) under the assumption $f(0) = 0$.

2. The general case $f(0) \neq 0$:
   We apply the above argument to the function $f(s) - f(0)$:

$$|f(s)| - |f(0)| \leq |f(s) - f(0)| \leq \frac{2r}{R-r} \sup_{|s|\leq R} \Re\{f(s) - f(0)\}$$
$$\leq \frac{2r}{R-r} \left(\sup_{|s|\leq R} \Re\{f(s)\} + |f(0)| \right), \tag{B.13}$$

   which leads to (B.5).

□

In 1912 the British mathematician John E. Littlewood (1885-1997) published a short article[6] on an estimate of $\log \zeta(s)$, related to the Lindelöf hypothesis (which will be discussed in a future report), in which he used the theorem, now known as the "Hadamard's three circles theorem," since it was published by Hadamard in 1896, but with no proof.

Theorem B.2 (Hadamard's Three-Circle Theorem)**.** *Consider three circles with radii $r_1 < r_2 < r_3$ all centered at the origin. Consider a holomorphic function $f(s)$ defined on the annulus $A(r_1, r_3) = \{s : r_1 \leq |s| \leq r_3\}$.*

Let $M(r_i)$ be the maximum of the modulus $|f(s)|$ on the circle of radius r_i. Then, the following inequality holds:

$$M(r_2) \leq M(r_1)^\theta M(r_3)^{1-\theta}, \quad \textit{where} \quad \theta = \frac{\log r_3 - \log r_2}{\log r_3 - \log r_1}, \tag{B.14}$$

If we take the logarithm,

$$\log M(r_2) \leq \theta \log M(r_1) + (1-\theta) \log M(r_3), \tag{B.15}$$

or equivalently,

$$\log\left(\frac{r_3}{r_1}\right) \log M(r_2) \leq \log\left(\frac{r_3}{r_2}\right) \log M(r_1) + \log\left(\frac{r_2}{r_1}\right) \log M(r_3). \tag{B.16}$$

In other words, $\log M(r)$ *is a convex function of* $\log r$.

Proof. Let a be a real number. Then the function $g(s)$ defined by

$$g(s) = a \log |s| + \log |f(s)|$$

is harmonic outside the zeros of $f(s)$. In the neighborhood of the zeros of $f(s)$, the function $g(s)$ assumes large values. From the maximum modulus principle, however, we know that $g(s)$ attains its maximum value on the boundary of $A(r_1, r_3)$. Therefore,

$$a \log |s| + \log |f(s)| \leq \max\{a \log r_1 + \log M(r_1), a \log r_3 + \log M(r_3)\} \ \text{ for all } \ s \in A(r_1, r_3). \tag{B.17}$$

Thus, by setting $|s| = r_2$ on RHS, and maximizing it, we have

$$a \log r_2 + \log M(r_2) \leq \max\{a \log r_1 + \log M(r_1), a \log r_3 + \log M(r_3)\}, \quad r_1 < r_2 < r_3. \tag{B.18}$$

Now let a be such that the two values inside $\max\{\ldots\}$ are equal, i.e.,

$$a = \frac{\log M(r_3) - \log M(r_1)}{\log r_1 - \log r_2}. \tag{B.19}$$

For this choice of a, the second term in the $\max\{\ldots\}$ in (B.18) is the same as the first term, we obtain

$$\log M(r_2) \leq a \log r_1 + \log M(r_1) - a \log r_2, \tag{B.20}$$

which, together with (B.19), leads to (B.15). □

[6] J. E. Littlewood (1912),"Quelques conséquences de l'hypothése que la fonction $\zeta(s)$ n'a pas de zéros dans le demi-plan $\Re(s) > \frac{1}{2}$," *Comptes Rendus de l'Acad. des Sciences (Paris)*, vol, 154, pp. 263-266.

In the aforementioned 1912 article, Littlewood applied this three-circle theorem and proved that if the Riemann hypothesis is true, then for every $\epsilon > 0$ and $\delta > 0$

$$|\log \zeta(\sigma + it)| < C(\log t)^{2-2\sigma+\epsilon} \quad \text{for} \quad \sigma \in [\tfrac{1}{2} + \delta, 1], \quad t \geq 2, \tag{B.21}$$

where C is a constant (see Edwards [3], p. 188) Since $\sigma > \frac{1}{2}$ implies $2 - 2\sigma + \epsilon < 1$ for sufficiently small ϵ, the Riemann hypothesis implies

$$\log |\zeta(\sigma + it)| = \Re\{\log \zeta(\sigma + it)\} \leq |\log \zeta(\sigma + it)| \leq K \log t (\log t)^{-\theta}, \tag{B.22}$$

where $\theta > 0$. Hence, for any $\epsilon' > 0$,

$$\log |\zeta(\sigma + it)| \leq \epsilon' \log t, \quad \text{for all sufficiently large } t. \tag{B.23}$$

Hence

$$|\zeta(s)| < t^{\epsilon'} \quad \text{for} \quad \Re(s) > \tfrac{1}{2}, \; \Im(s) = t \gg 1, \tag{B.24}$$

in short, the Riemann hypothesis implies the Lindelöf hypothesis, to be discussed in the next report.