No. 2: Riemann's Zeta Function $\zeta(s)$ and the Riemann Hypothesis

Towards a Proof of the Riemann Hypothesis

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Abstract

The Riemann zeta function and his famous conjecture regarding the property of this function were presented in his 1859 paper, which was concerned about the distribution of prime numbers. Thus, we begin with a brief historical review of earlier work by Gauss, Legendre and Chebyshev that later led to the prime number theorem (PNT). We then define Riemann's zeta function $\zeta(s)$ for $\sigma = \Re(s) > 1$, and then discuss how Riemann extended it to the entire s plane through analytic continuation. Then the Riemann hypothesis (RH) is explained, followed by a summary of important properties of the function $\zeta(s)$ together with a brief historical review of the progress on the subject. An extensive list of references on the subject is provided.
Key words: Distribution of prime numbers, Gauss' and Legendre's conjecture, Chebyshev's conjecture and the Chebyshev functions, Riemann's zeta function $\zeta(s)$, the gamma function $\Gamma(s)$, analytic continuation, Euler's product formula for $\zeta(s)$, Riemann hypothesis, properties of $\zeta(s)$.

1 Early Work on the Distribution of Prime Numbers

For a given positive number $x \geq 2$, the number of primes that are equal to or smaller than x is denoted $\pi(x)$, which is often referred to as the *prime counting function*. According to mathematical historians (see e.g., [5], pp. 53-55) the German mathematician Carl Friedrich Gauss (1777-1855)[20] conjectured around in 1792-73, when he was about 15-16 years old, that the logarithmic integral function

$$\mathrm{li}(x) = \int_0^x \frac{dt}{\log t} \tag{1}$$

would be a good approximation of $\pi(x)$. The function $\dfrac{1}{\log t}$ has a singularity at $t = 1$. Thus, for $x > 1$ the integral has to be interpreted as its *Cauchy principal value*:

$$\mathrm{li}(x) = \lim_{\epsilon \to 0} \left(\int_0^{1-\epsilon} \frac{dt}{\log t} + \int_{1+\epsilon}^x \frac{dt}{\log t} \right)$$

Because the function $1/\log t$ has a singularity at $t = 0$ (see Figure 1), an alternative form of the logarithmic integral is sometimes used:

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t} = \mathrm{li}(x) - \mathrm{li}(2) \approx \mathrm{li}(x) - 1.045163\ldots, \tag{2}$$

which is referred to as the *offset logarithmic integral* or *Eulerian logarithmic integral*. As shown in Figure 1, the function $\mathrm{li}(x)$ takes on zero at $x \approx 1.4513\ldots = \mu$. Then we can write

$$\mathrm{li}(x) = \int_\mu^x \frac{dt}{\log t}.$$

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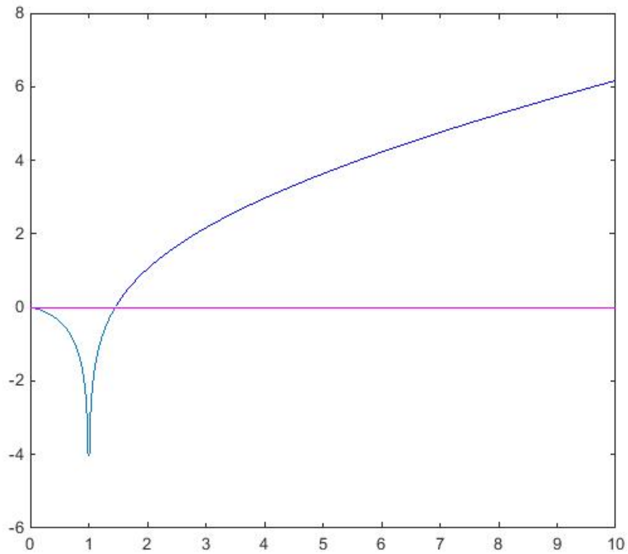

Figure 1: The logarithmic integral $\mathrm{li}(x)$

Figure 2: (a) Carl Friedrich Gauss (1777-1855), (b) Adrien-Marie Legendre (1752-1833)

Gauss did not bother to publish the result, but many years later, in 1849, he mentioned the above approximation formula in his correspondence with the astronomer Johann Franz Encke (1791-1865), who had studied mathematics under Gauss at the University of Göttingen.

The first published work on $\pi(x)$ came in 1798 from the the French mathematician Adrien-Marie Legendre (1752-1833)[1]. The approximation formula that Legendre proposed in his book entitled *Essay on the Theory of Numbers* published in 1798 was

$$\pi(x) \approx \frac{x}{A \log x + B}, \tag{3}$$

for some numbers A and B, "to be determined."[5]. In the second edition of the book published in 1808, he made a more precise conjecture with $A = 1$ and $B = -1.08366$ Both Legendre's and Gauss' results suggest the asymptotic formula

$$\pi(x) \sim \frac{x}{\log x}, \tag{4}$$

[1]The only known portrait of Legendre, shown in Figure 2, was recently found in the 1820 book of caricatures of seventy-three members of the Institut de France in Paris. See Kobayashi [10] pp. 40-42, and Wikipedia [21].

Figure 3: Pafnuty Chebyshev (1821-1894), Source: Wikipedia

which is known as "the asymptotic law of the distribution of prime numbers" and formed a basis of what is now known as *Prime Number Theorem* (PNT). Their conjecture was proved about 100 years later, independently by the French mathematician Jacques Hadamard (1865-1963) and the Belgian mathematician Charles-Jean de la Vallèe-Poussin (1866-1962): both published their work in 1869[2].

In the mean time the Russian mathematician Pafnuty Chebyshev (1821-1894) made significant contributions towards a proof of PNT. He first proved in his 1849 paper titled "On the function that determines the totality of prime numbers" that the limit

$$\lim_{x\to\infty}\frac{\pi(x)}{x/\log x} \tag{5}$$

if it exists, must be unity, although he could not show the existence of the limit. But he showed that there exist some positive constants such that

$$c_1\frac{x}{\log x} < \pi(x) < c_2\frac{x}{\log x}, \quad \text{for} \quad x \geq 2. \tag{6}$$

Furthermore, he derived the values of these constants for sufficiently large x, viz, $c_1 = 0.92129\ldots$, $c_2 = 1.10555\ldots$. To derive this result, he introduced two functions $\theta(n) = \sum_{p\leq n}\log p$ and $\psi(n) = \sum_{p^k\leq n}\log p$, which are crucial in later proofs of prime number theorem. We will postpone further discussion of the Chebyshev functions till a later chapter.

2 Bernhard Riemann and his seminal paper of 1859 [5, 16, 23]

Georg Friedrich Bernhard Riemann (1826-1866) first enrolled at the University of Göttingen in 1846 to study towards a degree in Theology (his father Friedrich Bernhard Riemann was a Lutheran minister), but he began studying mathematics under Carl Friedrich Gauss, who recommended Riemann to switch to the mathematical field. With his father's approval, he transferred to the University of Berlin in 1847, and studied under Carl Jacobi (1804-1851), Peter Gustav Lejeune Dirichlet (1805-1859), Jacob Steiner (1796-1863) and Gotthold Eisenstein (1823-52). The main person who influenced him at that time was Dirichlet. After a two years' stay in Berlin, Riemann returned to GÖttingen and his Ph.D. thesis on theory of complex variables supervised by Gauss was submitted in 1851. In his report on Riemann's thesis, Gauss described Riemann as having a gloriously fertile originality[16]. Riemann gave his first lectures in 1854, which founded the field of Riemannian geometry and thereby set the stage for Albert Einstein's general theory of relativity.

[2] Their proofs are called "analytic," in the sense they used complex function theory, especially properties of the Riemann zeta function[?]. Proofs that make no use of the zeta function or complex function theory are called "elementary." The first elementary proofs were discovered by the Norwegian mathematician Atle Selberg (1917-2007) and the Hungarian mathematician Paul Erdös (1913-1996), both in 1949.

In 1855 his mentor Gauss died, and Dirichlet succeeded him as a professor. In 1859, Dirichlet died and Riemann was promoted to the chair of mathematics at Göttingen. He was soon elected to the Berlin Academy of Sciences, recommended by three Berlin mathematicians; Kummer, Borchardt and Weierstrass. A newly elected member of the Academy had to report on their most recent research, and Riemann sent a report on "On the number of primes less than a given magnitude.

His 1859 seminal paper [15][3], on which our series of notes will focus, made pivotal contributions to modern analytic number theory. In this eight page article, the only paper by Riemann on the subject of number theory, he examined the zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right),$$

which was earlier investigated by Euler for cases where the argument s was restricted to real integers. But in Riemann's zeta function $s = \sigma + it$ is a complex number. Thus, $\zeta(s)$ now bears his name, and its importance was soon established in the study of the distribution of prime numbers.

Riemann's mathematical contributions are diverse, ranging from the aforementioned Riemannian geometry (e.g., Riemann surface, Riemann curvature tensor, Riemann metric), to complex analysis (e.g. Cauchy-Riemann equations, the Riemann mapping theorem, the Riemann-Roch theorem), real analysis (e.g., Riemann integral, Riemann-Stieltjes integral, Riemann-Lebesgue lemma), and number theory. Unfortunately he died on June 16th 1866 at age of 39 due to tuberculosis, while he was trying to recover at Selasca in northern Italy.

3 Riemann zeta function and Riemann hypothesis

3.1 Riemann zeta function ζ(s)

The Riemann zeta function $\zeta(s)$ was defined for $\Re(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1. \tag{7}$$

This function was then extended to the entire s-plane by analytic continuation, using an integral representation of $\zeta(s)$, as we shall discuss in the next section. The so-called Riemann hypothesis originated in this paper, where he stated in effect that all roots of the equation $\zeta(s) = 0$ have their real part to be $\frac{1}{2}$, except for "trivial zeros" which are located at even negative integers, $-2, -4, -6, \ldots$.

In (7), the argument [4] $s = \sigma + it$ is any complex number other than 1, and the value of $\zeta(s)$ is also complex. Riemann showed[5] that $\zeta(s)$ should satisfy the following *functional equation* in the strip $0 < \Re(s) < 1$:

$$\zeta(s) = 2(2\pi)^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s). \tag{8}$$

where $\Gamma(s)$ is the Gamma function defined by the following integral[6]

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t}\, dt. \tag{10}$$

[3]See the website of Cray Mathematical Institute (CMI) which posts Riemann's drafts [?].

[4]In most papers on the Riemann zeta function, it has been customary to write the complex variable as $s = \sigma + it$, instead of more common notation $z = x + iy$ or $s = x + iy$ used in complex analysis, in deference to Riemann who adopted this notation.

[5]Show e.g., Edwards [6] pp. 11-16. Titchmarsh [18] (pp. 13-44) provides a comprehensive treatment by presenting several different ways to derive the functional equation.

[6]The Gamma function in this form is also called the Euler integral of the second kind, and the beta function is called the Euler integral of the first kind;

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \Re(x) > 0,\ \Re(y) > 0. \tag{9}$$

The beta function was studied by Euler and Legendre.

It should be noted (see e.g.,[9] pp. 230-237) that Euler derived the following equation in 1749:

$$\frac{1-2^{n-1}+3^{n-1}-4^{n-1}+5^{n-1}-\cdots}{1-2^{-n}+3^{-n}-4^{-n}+5^{-n}-\cdots}=N(n)\frac{(n-1)!(2^n-1)}{(2^{n-1}-1)\pi^n}, \tag{11}$$

where $N(n)=0$, when n is odd, and $N(n)=(-1)^{m-1}$ when $n=2m$. He then conjectured that the following equation should hold for any s, although it is not clear whether he meant that s to be real or any complex number.

$$\frac{1-2^{s-1}+3^{s-1}-4^{s-1}+5^{s-1}-\cdots}{1-2^{-s}+3^{-s}-4^{-s}+5^{-s}-\cdots}=-\cos\frac{s\pi}{2}\frac{\Gamma(s)(2^s-1)}{(2^{s-1}-1)\pi^s}. \tag{12}$$

Equation (12) can be written in terms of $\zeta(s)$ as

$$\zeta(1-s)=\cos\frac{s\pi}{2}\Gamma(s)2^{1-s}\pi^{-s}\zeta(s), \tag{13}$$

where he uses the following identity between $\zeta(s)$ and the alternating series, which is later referred to as Dirichlet's eta function, denoted by $\eta(s)$.

$$(1-2^{1-s})\zeta(s)=\sum_{k=1}^{\infty}\frac{(-1)^k}{k^s}=\eta(s). \tag{14}$$

If we change $1-s$ to s in (13), we obtain

$$\zeta(s)=\sin\frac{s\pi}{2}\Gamma(1-s)2(2\pi)^{s-1}\zeta(1-s), \tag{15}$$

which is the functional equation (8) obtained by Riemann. It is amazing that Euler conjectured Riemann's functional equation 110 years earlier, when the theory of complex functions was yet to be developed.

3.2 Gamma function Γ(s)

The Gamma function[7] was introduced by Euler in 1729 as the following infinite product:

$$\Gamma(s)=\frac{1}{s}\prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^s}{\left(1+\frac{s}{n}\right)}=\lim_{n\to\infty}\frac{n!n^s}{s(s+1)\cdots(s+n)}. \tag{17}$$

This product representation and the Euler integral representation of (10) are equivalent for $\Re(s)>0$, as outlined below. By writing

$$e^{-t}=\lim_{n\to\infty}\left(1-\frac{t}{n}\right)^n,$$

and setting $\frac{t}{n}=u$, we have

$$\begin{aligned}\Gamma(s)&=\lim_{n\to\infty}\int_0^n t^{s-1}\left(1-\frac{t}{n}\right)^n dt=\lim_{n\to\infty}n^s\int_0^1 u^{s-1}(1-u)^n\,du\\&=\lim_{n\to\infty}n^s\int_0^1\frac{d}{du}\frac{u^s}{s}(1-u)^n\,du=\lim_{n\to\infty}n^s\frac{n}{s}\int_0^1 u^s(1-u)^{n-1}\,du,\end{aligned} \tag{18}$$

[7]Gauss introduced the notation in 1813

$$\Pi(s)=\int_0^\infty e^{-s}x^s\,dx,\quad s>-1. \tag{16}$$

Legendre subsequently introduced the notation $\Gamma(s)$ for $\Pi(s-1)$. Legendre's reasons for considering $(n-1)!$ instead of $n!$ are obscure. Edwards ([6] p. 8 footnote) notes that Legendre might have thought it was more natural to have the first pole occur at $s=0$ rather than $s=-1$. Legendre's notation prevailed in France and by the end of the nineteenth century, in the rest of the world as well. Riemann used Gauss' original notation, and Edwards [6] reintroduced it in his book.

where we used integration by part to arrive at the last expression. Repeating the same procedure $(n-1)$ more times, we will have

$$\Gamma(s) = \lim_{n\to\infty} n^s \frac{n(n-1)\cdots 1}{s(s+1)\cdots(s+n-1)} \int_0^1 u^{s+n-1}\, du = \lim_{n\to\infty} \frac{n! n^s}{s(s+1)\cdots(s+n)}, \tag{19}$$

which is (17). The integral form of the Gamma function can be extended by analytic continuation to $s \in \mathbb{C}$, all complex numbers, except for non-positive integers.

3.3 The Riemann hypothesis

The functional equation (8) allows us to define the zeta function for all remaining nonzero complex numbers s. The sine function in RHS has zeros at even integers. The Gamma function $\Gamma(s)$ has simple poles at all non-negative integers, thus $\Gamma(1-s)$ has poles at $s = 2, 3, 4, \ldots$. So the zeros of the sine functions at positive even integers are canceled by the poles of the Gamma function. Therefore, the zeta function has zeros at negative even integers: $s = -2, -4, -6, \ldots$, which are called the **trivial zeros**.

It is known, as we shall show later, that there are infinitely many **non-trivial zeros** and that they lie in the **critical strip** defined by $0 < \sigma < 1$. The Riemann hypothesis is concerned with the locations of the non-trivial zeros, and is stated as follows.

Conjecture 1 (Riemann Hypothesis). *The real part of any non-trivial zero of the zeta function $\zeta(s)$ is $\frac{1}{2}$.*

The line $\sigma = \frac{1}{2}$ in the complex plane is known as the **critical line**.

There are numerous technical and nontechnical books and articles on the Riemann hypothesis (see e.g., Borwein et al [2], Conrey [3], Edwards [6], Ivić [7], Iwaniec [8], Matsumoto [11], Mazur and Stein [12], Patterson [14] and Titchmarsh [18]). It has been empirically verified (see e.g., Odlyzko [13]) that over a billion zeros (out of infinitely many) all satisfy the Riemann hypothesis, but its mathematical proof is yet to be found. The Riemann hypothesis problem was put forth as one of the seven millennium prize problems[8]by the Cray Mathematics Institute in 2000 (see Bombieri [1] and Sarnak [17]).

The zeta function (7) defined for complex s is an absolutely convergent infinite series when its real part is greater than 1, i.e., $x = \Re\{s\} > 1$. As discussed in Report No. 1. (see Eqs. (60)) Leonhard Euler (1707-1783) showed that this series equals the product

$$\zeta(s) = \prod_{p:\text{prime}} \frac{1}{1-p^{-s}}, \tag{20}$$

where the infinite product extends over all prime numbers p, and converges for all s with real part greater than 1.

The Riemann hypothesis, however, is concerned with zeros outside the region of convergence of this series. Its analytic continuation to all complex number s can be done by using Dirichlet's eta function defined by

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \tag{21}$$

which is simply related to the zeta function as follows:

$$\eta(s) = \left(1 - 2^{1-s}\right) \zeta(s). \tag{22}$$

While the Dirichlet series expansion of the eta function $\eta(s)$ is convergent only for s with $\Re\{s\} > 0$, this series is Abel summable for any complex number. This result allows us to define the eta function as an entire function, and the relation (22) shows that the zeta function is meromorphic with a simple pole at $s = 1$ and poles at the zeros of $\left(1 - 2^{1-s}\right)$, which are $s = 1 + 2\pi i n/\ln 2$.

[8]The seven problems are: 1. Yang-Mills and mass gap, 2. Riemann hypothesis, 3. P. vs. NP problem, 4. Navier-Stokes equation, 5. Hodge conjecture, 6. Poincaré conjecture and 7. Birch and Swinnerton-Dyer conjecture. The sixth problem, Poincaré conjecture, was solved by the Russian mathematician Grigori Perelman (1966-) presented a proof of the conjecture in three papers avaiable in 2002 and 2003 on arXiv. Perelman's work survived review and was confirmed in 2006, leading to his being offered a Fields Medal, which he declined. Perelman was awarded the Millennium Prize on March 18, 2010. On July 1, 2010, he turned down the prize saying that he believed his contribution in proving the Poincaré conjecture was no greater than Hamilton's (who first suggested using the Ricci flow for the solution).

4 Some known properties of $\zeta(s)$

In this section we will give a brief summary of important known properties of $\zeta(s)$, with some historical accounts. Some of these properties will be further discussed in future reports.

1. Riemann's landmark paper of 1859 [15] began with the aforementioned relationship between $\zeta(s)$ and all primes (23), known as the *Euler product* formula, viz.

$$\prod_p \frac{1}{1-\frac{1}{p^s}} = \sum \frac{1}{n^s}, \quad \Re(s) > 1, \tag{23}$$

   where p ranges over all prime numbers, and n over all whole numbers. From this formula, we can see that $\zeta(s)$ has no zeros in the region $\Re(s) > 1$, because a convergent infinite product can be zero only if one of its factors is zero.

2. In the same paper Riemann showed that $\zeta(s)$ can be expressed in terms of a contour integral

$$\zeta(s) = \frac{i}{2\sin(\pi s)\Gamma(s)} \oint_C \frac{(-w)^{s-1}}{e^w - 1}\, dw, \tag{24}$$

   where C is a closed path that starts at $\infty - i0$, encircles $w = 0$, and returns to $\infty + i0$. The function $\zeta(s)$ thus defined is *holomorphic*[9] for all s except for a simple pole at $s = 0$.

3. Riemann also derived the aforementioned *functional equation* of $\zeta(s)$ (see e.g. [6] pp. 12-16):

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s), \quad \text{for all } s, \tag{25}$$

   or equivalently

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)\zeta(1-s), \quad \text{for all } s. \tag{26}$$

   By setting $s = 2n+1$ in (25), we readily find that $\zeta(-2n) = 0$, that is $\zeta(s)$ has zeros at $s = -2n, \ \ n = 1, 2, 3, \ldots$, which are aptly called *trivial zeros*. Recall that Riemann's conjecture is that all *non-trivial zeros* should be located on the straight line $s = \frac{1}{2} + it$.

4. By defining a complex-valued function

$$\xi(s) = \Gamma\left(\frac{s}{2}+1\right)(s-1)\pi^{-\frac{s}{2}}\zeta(s) \quad \text{for} \quad s = \sigma + it \tag{27}$$

   Riemann obtained the symmetric relation:

$$\xi(1-s) = \xi(s), \quad \text{for all } s. \tag{28}$$

   This implies $1-\rho$ is a root of $\xi(s)$ if and only if ρ is a root. Combining this observation with the preceding result based on the Euler product, one readily sees that $\zeta(s)$ has no zeros in the the region $\Re(s) < 0$. Consequently, all the nontrivial zeros must lie in the strip of $0 < x < 1$, the so-called "critical strip." We will further discuss the function $\xi(s)$ in Report No. 3.

5. In 1914, the British mathematician Godfrey H. Hardy (1877-1947) showed that infinitely many of the zeros are on the critical line $s = \frac{1}{2} + it$ See e.g., Edwards [6] pp. 226-229.

[9] A complex-valued function $f(s)$ is said to be *holomorphic* at a point $s = a$, if it is differentiable at every point within some open disk centered at $s = a$. The function $f(s)$ is said to be *analytic* at $s = a$ if in some open disk centered at $s = a$ it can be expanded as a convergent power series

$$f(s) = \sum_{n=0}^{\infty} c_n (s-a)^n.$$

One of the most important theorems of complex analysis is that holomorphic functions are analytic.

6. Carl Ludwig Siegel (1896-1981) discovered in the archives of the University Library of Göttingen Riemann's unpublished work ([6], p. 136). After having deciphered Riemann's handwritten notes with great difficulty, Siegel published a paper in 1932 [**?**], describing two topics, one on an asymptotic formula for the computation of $Z(t)$, which is now known as the Riemann-Siegel formula, and the other, a new representation of $\zeta(s)$ in terms of definite integrals.

7. The above mentioned function $Z(t)$ is defined by

$$Z(t) = \zeta\left(\tfrac{1}{2} + it\right) e^{i\vartheta(t)}, \tag{29}$$

where

$$\vartheta(t) = \Im \log\left(\Gamma\left(\tfrac{1}{4} + \tfrac{it}{2}\right)\right) - \tfrac{t}{2}\log \pi. \tag{30}$$

is called the Riemann-Siegel theta function. The function $Z(t)$ is real for real t and has the same zeros as $\zeta(s)$ at $s = \frac{1}{2} + it$, with t real. Thus, locating zeros on the critical line $\sigma = \frac{1}{2}$ reduces to locating zeros on the real line of the real function $Z(t)$. Furthermore, $|Z(t)| = |\zeta(\frac{1}{2} + it)|$.

8. The Riemann-Siegel formula gave the starting point for all large-scale calculations of $\zeta(s)$. So far all zeros that have been checked are on the critical line and are simple. Wikipedia [**?**] provides an extensive list concerning who (when) calculated how many zeros. Here are several noteworthy ones.

   - Riemann (1859?) calculated 3 zeros.
   - Titchmarsh (1935), 195 zeros, using the Riemann-Siegal formula.
   - Titchmarsh and Comrie (1936), 1041 zeros. They were the last to find zeros by hand.
   - Turing (1953), 1104 zeros. He was the first to use a digital computer.
   - Lehmer (1956), 25,000 zeros.
   - Lehman (1966), 250,000 zeros.
   - Brent (1979), 81,000,001 zeros.
   - van de Lune,te Riele & Winter (1986), 1,500,000,001 zeros. Gave several graphs of Z at places of unusual behavior.
   - Odlyzko (1992), 175 million zeros of heights around $t = 10^{20}$ and a few more of heights around $t = 2 \times 10^{20}$ [**?**].
   - Wedeniwski (2004), 900 billion zeros. Used ZetaGrid distributed computing, which was terminated in 2005. `http://en.wikipedia.org/wiki/ZetaGrid`
   - Gourdon and Demichel (2004), 10 trillion zeros and a few of large (up to $t \sim 10^{24}$) heights. Used the Odlyzko-Scönhage algorithm.

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