No. 4: Fourier Transform Representation of the Function $\xi(s)$

Towards a Proof of the Riemann Hypothesis

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Abstract

Starting from the integral representation for $\xi(s)$ given by Riemann, we obtain a Fourier transform representation of $\xi(s)$ in terms of a kernel function, denoted $S(\omega)$. Then we show that the Fourier transform of $S(\omega)$ into the "complex time" domain $\tau = t - i\lambda$ (where $s = \sigma + it = \frac{1}{2} + \lambda + it$) yields $\xi(s)$. For $\lambda = 0$, the function $\Xi(t) = \xi(\frac{1}{2} + it)$ and $S(\omega)$ form the Fourier transform pair à la the Wiener-Khinchin theorem.

The kernel $S(\omega)$ can be represented as $S(\omega) = \sum_{n=1}^{\infty} S_n(\omega)$, and the Fourier transform representation proposed by George Pólya is based on the argument that $S(\omega) \approx S_1(\omega)$.

We show that the kernel function $S(\omega)$ is symmetric, i.e., $S(-\omega) = S(\omega)$, thus we can derive an alternative Fourier representation of $\Xi(t)$ in terms of $S(\omega)$; $-\infty < \omega < \infty$. We then show that the Fourier transform of $S_1(\omega)$ is equal to $g(t)$ of (3) in Report No. 3, whose multiplication to the zeta function $\zeta(s)$ leads to the $\xi(s)$ function.

Key words: Fourier transform, kernel function $S(\omega)$, Wiener-Khinchin theorem, George Pólya, symmetry of the kernel $S(\omega)$, "Real time variable" t versus "complex time," Fourier transform pair $g(t)$ and $S_(\omega)$.

1 The Fourier transform representation of $\xi(s)$

1.1 Riemann's integral representation of ξ(s).

Riemann [5] derived the following integral representation of $\xi(s)$ (see [1], pp. 16-18).

$$\xi(s) = \frac{1}{2} - \frac{s(1-s)}{2} \int_1^{\infty} \psi(x) \left(x^{s/2} + x^{(1-s)/2}\right) \frac{dx}{x}, \tag{1}$$

where

$$\psi(x) = \sum_{n=1}^{\infty} e^{-n^2 \pi x} \tag{2}$$

is called the *theta function*. By applying integration by parts to (1) and Jacobi's identity for the theta function[1] Edwards ([1], p. 17) gives the following expression by generalizing Riemann's result, which holds for any complex number s, not just for the values on the critical line:

$$\xi(s) = 4 \int_1^{\infty} \frac{d[x^{3/2}\psi'(x)]}{dx} x^{-1/4} \cosh\left[\tfrac{1}{2}\left(s - \tfrac{1}{2}\right)\log x\right] \, dx. \tag{4}$$

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[1] Jacobi's identity for the theta function $\psi(x)$ is

$$2\psi(x) + 1 = x^{-1/2}\left(2\psi(x^{-1}) + 1\right). \tag{3}$$

By writing

$$\frac{d[x^{3/2}\psi'(x)]}{dx}x^{-1/4} = \pi x^{1/4} D(x) \tag{5}$$

with $D(x)$ defined by

$$D(x) = \sum_{n=1}^{\infty} n^2(n^2\pi x - \tfrac{3}{2})e^{-n^2\pi x} > 0, \quad \text{for} \quad x \geq 1, \tag{6}$$

we can write (4) as

$$\xi(s) = 4\pi \int_1^\infty x^{1/4} D(x) \cos\left(\frac{\tau \log x}{2}\right) dx, \tag{7}$$

where

$$\tau = t - i\lambda = -i(s - \tfrac{1}{2}), \tag{8}$$

and we used the identity $\cosh(iy) = \cos y$. By changing the variable from x to ω by

$$\omega = \frac{\log x}{2}. \tag{9}$$

and defining

$$S(\omega) = 8\pi e^{5\omega/2} D(e^{2\omega}), \tag{10}$$

we can write (7) as

$$\xi(s) = \int_0^\infty S(\omega)\cos(\omega\tau)\, d\omega. \tag{11}$$

On the critical line $s = \frac{1}{2} + it$ (i.e., when $\lambda = 0$), the above reduces to a more familiar formula

$$\Xi(t) = \int_0^\infty S(\omega)\cos(\omega t)\, d\omega. \tag{12}$$

1.2 The kernel function S(ω) as a power spectral function.

The kernel $S(\omega)$ is positive for all $\omega \geq 0$, because $D(x)$ is positive for $x \geq 1$. Therefore, $S(\omega)$ can qualify as a *spectral density function* of a certain wide-sense stationary process, and we can interpret $\Xi(t)$ as its autocorrelation function (see e.g., [3] p. 349). In this context, the transforms between the spectrum $S(\omega)$ and the function $\Xi(t)$ are what is known as the Wiener-Khinchin formula. The inverse transform to (12), to be given by (16), exists under a very general condition.

The Fourier transform representation (12) has been studied by George Pólya [4] and others (see e.g., Titchmarsh [6], Chapter 10). Dimitrov and Rusev [2] extensively review the past work on "zeros of entire Fourier transforms," including Pólya's work in the 1920s. Many researchers have investigated the Riemann hypothesis along the line of finding a necessary and sufficient condition that the kernel must satisfy in order for its Fourier transform to have only real zeros. To the best of the author's knowledge, however, no such condition has been found yet.

From the above observation that $S(\omega)$ is positive for $\omega > 0$, we can readily establish the following proposition.

Theorem 1.1. *The modulus $|\Xi(t))|$ is maximum at $t = 0$, i.e.,*

$$|\Xi(t)| \leq \Xi(0) = 0.4971\ldots, \quad \textit{for all} \quad t. \tag{13}$$

Furthermore,

$$\int_0^\infty \Xi(t)\, dt = 3\pi\left(\frac{\pi^{1/4}}{\Gamma(3/4)} - 1\right) = 2.8067\ldots. \tag{14}$$

Figure 1: George Pólya (1887-1985): source: California Mathematics Council http://cmc-math.org/members/infinity/polya.html

Proof. From (11), it readily follows that

$$|\Xi(t)| \leq \int_0^\infty |S(\omega)|\, d\omega = \int_0^\infty S(\omega)\, d\omega = \Xi(0). \tag{15}$$

Since $\zeta(\frac{1}{2}) = -1.46035\ldots$[2], and $g(\frac{1}{2}) = -\frac{1}{8}\pi^{-1/4}\Gamma(\frac{1}{4}) = -0.3404\ldots$, we have $\Xi(0) = \xi(\frac{1}{2}) = g(\frac{1}{2})\zeta(\frac{1}{2}) = 0.4971\ldots$.

From the Wiener-Khinchin inverse formula (see e.g., [3], p. 349), we have

$$S(\omega) = \frac{2}{\pi}\int_0^\infty \Xi(t)\cos(\omega t)\, dt. \tag{16}$$

By setting $\omega = 0$, we readily find

$$S(0) = \frac{2}{\pi}\int_0^\infty \Xi(t)\, dt. \tag{17}$$

By setting $\omega = 0$ in (10), we have

$$S(0) = 8\pi D(1) = 8\left(\tfrac{3}{2}\psi'(1) + \psi''(1)\right). \tag{18}$$

The function $\psi(x)$ satisfies the aforementioned Jacobi's identity (3). By differentiating the identity equation, we find

$$2\psi'(x) = -\tfrac{1}{2}x^{-3/2} - x^{-3/2}\psi(1/x) - 2x^{-5/2}\psi'(1/x) \tag{19}$$

By setting $x = 1$ in (19) we obtain

$$\psi'(1) = -\tfrac{1}{8}(1 + 2\psi(1)) \tag{20}$$

$$\tag{21}$$

The value of $\psi(1)$ is known (see e.g., Yi [7], Theorem 5.5 in p. 398)

$$\psi(1) = \tfrac{1}{2}\left(\frac{\pi^{1/4}}{\Gamma(3/4)} - 1\right) = \tfrac{1}{2}\left(\frac{1.3313}{1.2254} - 1\right) = 0.0432\ldots. \tag{22}$$

Hence,

$$\psi'(1) = -\tfrac{1}{8}\frac{\pi^{1/4}}{\Gamma(3/4)} = -0.1358\ldots. \tag{23}$$

[2]See e.g. https://oeis.org/A059750.

Unfortunately, $\psi''(1)$ cannot be easily found, because when we differentiate (19) once more and set $x = 1$, the $\psi''(1)$ terms appear in both sides and are canceled out. But its numerical evaluation is straightforward, since the series rapidly converges,

$$\psi''(1) = \pi^2 \sum_{n=1}^{\infty} n^4 e^{-\pi n^2} \approx \pi^2 \sum_{n=1}^{2} n^4 e^{-\pi n^2} = 0.4271\ldots. \tag{24}$$

Thus, we finally evaluate

$$\int_0^\infty \Xi(t)\, dt = \frac{\pi}{2} S(0) = 4\pi \left(\tfrac{3}{2}\psi'(1) + \psi''(1)\right) = 2.8067\ldots \tag{25}$$

□

Remark: "Real time" t vs, "complex-time" τ.

The variable t of the complex variable $s = \sigma + it = \frac{1}{2} + \lambda + it$ is the imaginary component of the complex variable s, and is often called the *height* in the zeta function related literature, especially when we speak of the critical line, for which $\sigma = \frac{1}{2}$ or the critical strip, where $0 < \sigma < 0$. Here it may be appealing to physicists and engineers, in the context of the Fourier transform representation, to interpret t as "time" and the variable ω of the kernel function $S(\omega)$ as the "(angular) frequency." Then, it may be appropriate to refer to the complex number τ defined by (8), i.e., $\tau = t - i\lambda = -i\left(s - \frac{1}{2}\right)$ as "complex-time." Use of the complex-time τ allow us to write compactly, as given earlier in (11)

$$\xi(s) = \int_0^\infty S(\omega) \cos(\omega\tau)\, d\omega. \tag{26}$$

This interpretation of Riemann's result (4) will shed some new light to the Fourier transform representation of the $\xi(s)$ function. We will further discuss this in a later section.

2 Further characterization of the function ξ(s)

2.1 Decomposition of S(ω)

In the Fourier transform representation (11) the kernel function $S(\omega)$ can be expressed as

$$S(\omega) = \sum_{n=1}^{\infty} S_n(\omega), \tag{27}$$

with

$$S_n(\omega) = 8\pi e^{5\omega/2} D_n(e^{2\omega}), \tag{28}$$

where

$$D_n(x) = n^2(n^2\pi x - \tfrac{3}{2}) e^{-n^2\pi x}. \tag{29}$$

The Fourier transform can therefore be written as a summation of infinite components, i.e.,

$$\xi(s) = \sum_{n=1}^{\infty} f_n(s), \tag{30}$$

with

$$\begin{aligned} f_n(s) &= \int_0^\infty S_n(\omega) u(\omega) \cos(\omega\tau)\, d\omega \\ &= 8\pi \int_0^\infty e^{5\omega/2} D_n(e^{2\omega}) \cos(\omega\tau)\, d\omega, \end{aligned} \tag{31}$$

where τ is the "complex time" defined by (8). The above exchange in the order of taking summation over n and taking the cosine transform of $S_n(\omega)u(\omega)$, as shown in (31) and (30), should be justifiable, since the series $S_n(\omega)u(\omega)$ converges to $S(\omega)u(\omega)$ for the entire $\omega \geq 0$.

The kernel $S_n(\omega)$ of (28) can be written as

$$S_n(\omega) = 8\pi n^2 e^{\frac{5\omega}{2}} D_1(n^2 e^{2\omega}), \tag{32}$$

with

$$D_1(x) = (\pi x - \tfrac{3}{2})e^{-\pi x}. \tag{33}$$

We can write $S_n(\omega)$ in terms of $S_1(\omega)$ as follows:

$$S_n(\omega) = \frac{1}{\sqrt{n}} S_1(\omega + \log n). \tag{34}$$

In the range $\omega \geq 0$, $S(\omega)$ is largely determined by its first few components $S_n(\omega)$, $n = 1, 2, \ldots$, since the series (27) rapidly converges in $\omega \geq 0$. But they behave in an entirely different manner in the range $\omega < 0$.

2.2 The Fourier transform of S(ω) in $-\infty < \omega < \omega$.

In this section we consider the Fourier transform of $S(\omega)$, $-\infty < \omega < \infty$, which will give us a further insight. First, note that the kernel $S(\omega)$ used in (11) is symmetric, i.e.,

$$S(-\omega) = S(\omega), \quad -\infty < \omega < \infty, \tag{35}$$

which can be shown as follows. Let us denote the expression of (5) as $C(x)$:

$$C(x) = \pi x^{1/4} D(x) = \frac{d\left[x^{3/2}\psi(x)\right]}{dx} x^{-1/4} \tag{36}$$

Then, the kernel function $S(\omega)$ of (10) can be written as

$$S(\omega) = 8e^{2\omega} C(e^{2\omega}). \tag{37}$$

Now we wish to prove (35). By setting $x = y^{-1}$ in (36), we have

$$C(y^{-1}) = \frac{d\left[y^{-3/2}\psi'(y^{-1})\right]}{dy^{-1}} y^{1/4} = \tfrac{1}{2} y^{-5/4}\left[2\psi''(y^{-1}) + 3y\psi'(y^{-1})\right]. \tag{38}$$

By differentiating Jacobi's identity (3), we find

$$\begin{aligned} \psi'(y^{-1}) &= -y^{5/2}\psi'(y) - \tfrac{1}{2}y^{3/2}\psi(y) - \tfrac{1}{4}y^{3/2} \\ \psi''(y^{-1}) &= y^{9/2}\psi''(y) + 3y^{7/2}\psi'(y) + \tfrac{3}{4}y^{5/2}\psi(y) + \tfrac{3}{8}y^{5/2}. \end{aligned} \tag{39}$$

Then

$$2\psi''(y^{-1}) + 3y\psi'(y^{-1}) = y^{7/2}\left[2y\psi''(y) + 3\psi'(y)\right]. \tag{40}$$

By substituting this into (38), we obtain

$$C(y^{-1}) = \tfrac{1}{2} y^{1/4}\left[2y\psi''(y) + 3\psi'(y)\right] y^2 = C(y)y^2. \tag{41}$$

Thus, by setting $x = e^{2\omega}$ in (**??**)

$$S(-\omega) = 8x^{-1}C(x^{-1}) = 8x^{-1}C(x)x^2 = 8xC(x) = S(\omega). \tag{42}$$

Thus, the symmetric property (35) has been proved.

The Fourier transform representation (11) can be changed to

$$\xi(s) = \tfrac{1}{2}\int_{-\infty}^{\infty} S(\omega)e^{i\omega\tau}\,d\omega = \tfrac{1}{2}\int_{-\infty}^{\infty} S(\omega)e^{\omega(s-\frac{1}{2})}\,d\omega. \tag{43}$$

Since the kernel $S(\omega)$ is a symmetric real function, we can readily derive the reflective property $\xi(1-s) = \xi(s)$ and thus $\xi(s)$ is real on the critical line. By substituting (27) and (34) into the above, we obtain

$$\begin{aligned}\xi(s) &= \tfrac{1}{2}\int_{-\infty}^{\infty}\sum_{n=1}^{\infty} S_n(\omega)e^{i\omega\tau}\,d\omega = \tfrac{1}{2}\int_{-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}S_1(\omega+\log n)e^{i\omega\tau}\,d\omega\\ &= \tfrac{1}{2}\int_{-\infty}^{\infty} S_1(\omega')e^{i\omega'\tau}\,d\omega'\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}e^{-i\tau\log n},\end{aligned} \tag{44}$$

where we set $\omega + \log n = \omega'$ in the above derivation. The summed term is nothing but the zeta function $\zeta(s)$, i.e.,

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}e^{-i\tau\log n} = \sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}+i\tau}} = \zeta\left(\tfrac{1}{2}+i\tau\right) = \zeta(s), \tag{45}$$

We denote the Fourier transform of $S_n(\omega)$ over $-\infty < \omega < \infty$ as $\xi_n(s)$:

$$\xi_n(s) = \tfrac{1}{2}\int_{-\infty}^{\infty} S_n(\omega)e^{i\omega\tau}\,d\omega. \tag{46}$$

Note that the functions $\xi_n(s)$ are individually complex functions even on the critical line, since $S_n(\omega)$ are not symmetric functions, thus $\xi_n(s)$'s do not satisfy the reflective property that their sum $\xi(s)$ does.

The result (44) can be compactly expressed as

$$\xi(s) = \xi_1(s)\zeta(s). \tag{47}$$

By referring to the definition of $\xi(s)$ given by Eq. (2) of Report No.3, we find from (47) that $\xi_1(s)$ should be equal to $g(s)$ defined by Eq.(3) of Report No. 3, viz.

$$\xi_1(s) = g(s) = \frac{s(s-1)}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right). \tag{48}$$

The results (47) and (48) are new to the best of the author's knowledge. The above observation leads us to the following proposition:

Theorem 2.1. *(The Fourier transform of* $\mathbf{S_1}(\omega)$*)*
The function $g(s)$ *defined by (3) of Report No. 3 is the Fourier transform of* $S_1(\omega)$ *to the complex-time domain* τ*, i.e.,*

$$g(s) = \tfrac{1}{2}\int_{-\infty}^{\infty} S_1(\omega)e^{i\omega\tau}\,d\omega, \tag{49}$$

where $\tau = t - i\lambda = t - i(\sigma - \frac{1}{2}) = -i(s - \frac{1}{2})$.

Proof. Let the right hand side of (49) be

$$I(s) = \tfrac{1}{2}\int_{-\infty}^{\infty} S_1(\omega)e^{i\omega\tau}\,d\omega, \quad \text{where} \quad \tau = t - i\left(s - \tfrac{1}{2}\right). \tag{50}$$

By substituting

$$S_1(\omega) = 8\pi e^{5\omega/2}(\pi e^{2\omega} - \tfrac{3}{2})e^{-\pi e^{2\omega}}, \tag{51}$$

into (50) and setting $\pi e^{2\omega} = y$, we obtain

$$\begin{aligned} I(s) &= 4\pi \int_{-\infty}^{\infty} e^{5\omega/2}(\pi e^{2\omega} - \tfrac{3}{2}) e^{-\pi e^{2\omega}} e^{\omega(s-\frac{1}{2})}\, d\omega \\ &= 2\pi^{-\frac{s}{2}} \int_0^\infty e^{-y} y^{\frac{s}{2}+1}\, dy - 3\pi^{-\frac{s}{2}} \int_0^\infty e^{-y} y^{\frac{s}{2}}\, dy \\ &= \pi^{-\frac{s}{2}} \left[2\Gamma\left(\tfrac{s}{2}+2\right) - 3\Gamma\left(\tfrac{s}{2}+1\right) \right] \\ &= \tfrac{1}{2} s(s-1)\pi^{-\frac{s}{2}} \Gamma\left(\tfrac{s}{2}\right) = g(s). \end{aligned} \tag{52}$$

Thus, we have proved (49). □

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