No. 3: Riemann's xi Function $\xi(s)$ and Its product form representation

Towards a Proof of the Riemann Hypothesis

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Abstract

In his 1859 paper, Riemann introduced the $\xi(s)$ function of the form $\xi(s) = g(s)\zeta(s)$, where $g(s)$ is chosen so that $\xi(s)$ satisfies the reflective property, i.e, $\xi(1-s) = \xi(s)$, which implies that $\xi(\frac{1}{2}+it) = \Xi(t)$ is a real function of t.

We then derive the relations between the log derivatives of a holomorphic function and the log partial derivatives of the the modulus of the holomorphic function. The product form representation of $\xi(s)$ obtained by Hadamard using the Weierstrass' factorization theorem can be reduced to the simpler product form conjectured by Riemann. We then show that the Riemann hypothesis is equivalent to the monotone decreasing property of the modulus $|\xi(\sigma + it)|$ in $-\infty < \sigma < \frac{1}{2}$ for given t.

In the final section, we derive relations between the $\xi(s)$'s partial derivatives w.r.t σ and t, which will be used later.

Key words: Reflective property of $\xi(s)$, Hadamard's product formula, Riemann's conjecture for the product formula, log derivatives, digamma function, Monotonicity of $|\xi(s)|$, function $\Xi(s)$, function $a(t)$.

1 Introduction: Function $\xi(s)$ and its properties

As we discussed in Report No. 2, the Riemann zeta function $\zeta(s)$ was defined in his semina paper [7] of 1859 as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1, \tag{1}$$

which was extended by Riemann himself to the entire s-plane by analytic continuation, using some integral representation of $\zeta(s)$.

In this paper we investigate the function $\xi(s)$, which was introduced also by Riemann in the same paper (see e.g.,[1], pp. 16-18):

$$\xi(s) = g(s)\zeta(s), \tag{2}$$

where

$$g(s) = \frac{s(s-1)}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right) = (s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}+1\right), \tag{3}$$

where $\Gamma(s)$ is the gamma function defined in Report No. 2, equation (10). The function $\Gamma(s)$ has poles at negative integers, -1, -2, -3, ... Thus, $\Gamma(s/2)$ in (3) removes the aforementioned trivial zeros of $\zeta(s)$. The function $g(s)$ does not have any zeros, thus. the set of zeros of $\xi(s)$ is equivalent to the set of "nontrivial

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zeros" of $\zeta(s)$. As we discussed in Report No.2, Section 4, Items 1, 4, and 5, we know that there are countably infinitely many zeros. We enumerate them as $\rho_n = \sigma_n + it_n$. Then we know that $0 < \sigma_n < 1$ for all n. The critical area $\{(\sigma, t) : 0 < \sigma < 1, -\infty < t < \infty\}$ is called the *critical strip*, and the line $\sigma = \frac{1}{2}$ is called the *critical line*.

The reason why we investigate the function $\xi(s)$ is that it is more convenient to deal with than $\zeta(s)$, since it is "reflective" in the sense

$$\xi(1-s) = \xi(s). \tag{4}$$

If we write $s = \frac{1}{2} + \lambda + it$, instead of $s = \sigma + it$, the critical line is defined by $\lambda = 0$. The real and imaginary parts of $\xi(s)$ satisfy the following properties:

$$\Re\left\{\xi\left(\tfrac{1}{2} + \lambda + it\right)\right\} = \Re\left\{\xi\left(\tfrac{1}{2} - \lambda + it\right)\right\}, \tag{5}$$

$$\Im\left\{\xi\left(\tfrac{1}{2} + \lambda + it\right)\right\} = -\Im\left\{\xi\left(\tfrac{1}{2} - \lambda + it\right)\right\}, \tag{6}$$

By setting $\lambda = 0$ in (6), we find

$$\Im\left\{\xi\left(\tfrac{1}{2} + it\right)\right\} = 0, \quad \text{for all} \quad t, \tag{7}$$

which implies that $\xi(s)$ is real on the critical line, and the Riemann hypothesis is equivalent to stating that all roots of

$$\xi\left(\tfrac{1}{2} + it\right) = 0 \tag{8}$$

are real. It is indeed in terms of the locations of zeros of this equation that Riemann stated his conjecture[1].

From (7) and Laplace's equation[2] we readily find

$$\frac{\partial^2 \Im\left\{\xi\left(\frac{1}{2} + it\right)\right\}}{\partial \sigma^2} = -\Im\left\{\xi''\left(\tfrac{1}{2} + it\right)\right\} = 0. \tag{9}$$

Thus, $\Im\left\{\xi\left(\frac{1}{2} + \lambda + it\right)\right\}$ must be a polynomial in λ of degree 1 in the vicinity of $\lambda = 0$.

$$\Im\left\{\xi\left(\tfrac{1}{2} + \lambda + it\right)\right\} = b(t)\lambda, \quad \text{for } \lambda \approx 0. \tag{10}$$

Similarly, we can characterize $\Re\left\{\xi\left(\frac{1}{2} + \lambda + it\right)\right\}$ by a polynomial in λ of degree 2:

$$\Re\left\{\xi\left(\tfrac{1}{2} + \lambda + it\right)\right\} = \frac{a(t)}{2}\lambda^2 + \xi\left(\tfrac{1}{2} + it\right), \quad \lambda \approx 0. \tag{11}$$

2 Some mathematical preliminaries

2.1 Logarithmic Differentials of Complex Functions

We begin with the following lemma that is applicable to any holomorphic function.

[1] "..., und es ist sehr wahrscheinlich, dass alle Wurzeln reell sind. Hiervon wäre allerdings ein strenger Beweis zu wünschen; ich habe indess die Aufsuchung desselben nach einigen flüchtigen vergeblichen Versuchen vorläufig bei Seite gelassen, da er für den nächsten Zweck meiner Untersuchung entbehrlich schien." (`http://www.claymath.org/sites/default/files/zeta.pdf`). An English translation is: " ... and it is very probable that all roots are real. Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation." (`http://www.claymath.org/sites/default/files/ezeta.pdf`).

[2] Both real and imaginary parts of an analytic complex function $f(s)$, $s = \sigma + it$, satisfy the following equation

$$\frac{\partial^2 \Re\{f(s)\}}{\partial \sigma^2} + \frac{\partial^2 \Re\{f(s)\}}{\partial t^2} = 0, \quad \frac{\partial^2 \Im\{f(s)\}}{\partial \sigma^2} + \frac{\partial^2 \Im\{f(s)\}}{\partial t^2} = 0,$$

which is called Laplace's equation, named after the French mathematician Piere-Simon Laplace (1749-1827).

Lemma 2.1. *For any holomorphic function $f(s)$ such that $f(s) \neq 0$, we have*

$$\frac{1}{|f(s)|} \cdot \frac{\partial |f(s)|}{\partial \sigma} = \Re\left\{\frac{f'(s)}{f(s)}\right\}, \tag{12}$$

$$\frac{1}{|f(s)|} \cdot \frac{\partial |f(s)|}{\partial t} = -\Im\left\{\frac{f'(s)}{f(s)}\right\}, \tag{13}$$

where $f'(s) = \dfrac{df(s)}{ds}$.

Proof. The logarithm of $f(s)$ can be written as

$$\log f(s) = \log |f(s)| + i \arg(f(s)), \tag{14}$$

where both $|f(s)|$ and $\arg(f(s))$ are real numbers. By differentiating both sides w.r.t. σ, we have

$$\frac{\partial s}{\partial \sigma}\frac{f'(s)}{f(s)} = \frac{1}{|f(s)|}\frac{\partial |f(s)|}{\partial \sigma} + i\frac{\partial \arg(f(s))}{\partial \sigma}. \tag{15}$$

By taking the real part of both sides, and noting that $\dfrac{\partial s}{\partial \sigma} = 1$, we obtain (12).

Similarly, take the partial derivative of (14) w.r.t. t:

$$\frac{\partial s}{\partial t}\frac{f'(s)}{f(s)} = \frac{1}{|f(s)|}\frac{\partial |f(s)|}{\partial t} + i\frac{\partial \arg(f(s))}{\partial t}. \tag{16}$$

By noting that $\dfrac{\partial s}{\partial t} = i$, and by taking the real part of both sides, we obtain (13). □

Corollary 2.1. *For the holomorphic function $f(s)$ of Lemma 2.1 the following identities hold:*

$$\frac{1}{|f(s)|}\frac{\partial^2 |f(s)|}{\partial \sigma^2} - \left(\frac{1}{|f(s)|}\frac{\partial |f(s)|}{\partial \sigma}\right)^2 = \Re\left\{\frac{f''(s)}{f(s)} - \left(\frac{f'(s)}{f(s)}\right)^2\right\}, \tag{17}$$

$$\frac{1}{|f(s)|}\frac{\partial^2 |f(s)|}{\partial t^2} - \left(\frac{1}{|f(s)|}\frac{\partial |f(s)|}{\partial t}\right)^2 = -\Re\left\{\frac{f''(s)}{f(s)} - \left(\frac{f'(s)}{f(s)}\right)^2\right\}. \tag{18}$$

where $f''(s) = \dfrac{d^2 f(s)}{ds^2}$.

Proof. We can rewrite (15) as follows:

$$\frac{f'(s)}{f(s)} = \frac{1}{|f(s)|}\frac{\partial |f(s)|}{\partial \sigma} + i\frac{\partial \arg(f(s))}{\partial \sigma}. \tag{19}$$

We differentiate the above w.r.t. σ again, obtaining

$$\frac{f''(s)}{f(s)} - \left(\frac{f'(s)}{f(s)}\right)^2 = \frac{1}{|f(s)|}\frac{\partial^2 |f(s)|}{\partial \sigma^2} - \left(\frac{1}{|f(s)|}\frac{\partial |f(s)|}{\partial \sigma}\right)^2 + i\frac{\partial^2 \arg(f(s))}{\partial \sigma^2}. \tag{20}$$

By taking the real part of both sides, we obtain (17).

Similarly, we rewrite (16) as

$$i\frac{f'(s)}{f(s)} = \frac{1}{|f(s)|}\frac{\partial |f(s)|}{\partial t} + i\frac{\partial \arg(f(s))}{\partial t}. \tag{21}$$

Differentiate the above w.r.t. t again:

$$-\frac{f''(s)}{f(s)} + \left(\frac{f'(s)}{f(s)}\right)^2 = \frac{1}{|f(s)|}\frac{\partial^2 |f(s)|}{\partial t^2} - \left(\frac{1}{|f(s)|}\frac{\partial |f(s)|}{\partial t}\right)^2 + i\frac{\partial^2 \arg(f(s))}{\partial t^2}. \tag{22}$$

By taking the real part of both sides, we obtain (18). Alternatively, this can be obtained from (17) and Laplace's equation applied to $\log |f(s)|$. □

2.2 The Product Formula for $\xi(s)$

The following product form representation of $\xi(s)$ is ascribed to the French mathematician Jacques Hadamard (1865-1963) [2], who obtained the formula in 1893 based on Karl Weierstrass (1815-1897)'s factorization theorem[3].

$$\xi(s) = \tfrac{1}{2} e^{Bs} \prod_n \left[\left(1 - \frac{s}{\rho_n}\right) e^{s/\rho_n} \right], \tag{23}$$

where the product is taken over all (infinitely many) zeros ρ_n's of the function $\xi(s)$, and B is a real constant, which will be determined below. Detailed accounts of this formula are found in many books (see e.g., Patterson [6] and Iwaniec [4]). Sondow and Dumitrescu [8] and Matiyasevich et al. [5] explored the use of the above product form towards a possible proof of the Riemann hypothesis.

By taking the logarithm of (23) and differentiating it, we obtain

$$\frac{\xi'(s)}{\xi(s)} = B + \sum_n \left(\frac{1}{s-\rho_n} + \frac{1}{\rho_n} \right). \tag{24}$$

From the definition of $\xi(s)$ in (2), we have

$$\frac{\xi'(s)}{\xi(s)} = \frac{g'(s)}{g(s)} + \frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - \tfrac{1}{2}\log\pi + \tfrac{1}{2}\Psi\left(\frac{s}{2}+1\right) + \frac{\zeta'(s)}{\zeta(s)}, \tag{25}$$

where $\Psi(s)$ is the logarithmic derivative of $\Gamma(s)$:

$$\Psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}, \tag{26}$$

which is called the digamma function. We now equate (30) to (25) and set $s = 0$, obtaining

$$B + \sum_n \left(-\frac{1}{\rho_n} + \frac{1}{\rho_n} \right) = -1 - \tfrac{1}{2} + \tfrac{1}{2}\Psi(1) + \frac{\zeta'(0)}{\zeta(0)} \tag{27}$$

By using $\xi(0) = -\zeta(0) = \frac{1}{2}$, $\zeta'(0)/\zeta(0) = \log(2\pi)$, and $\Psi(1) = \Gamma'(1) = -\gamma$ (where γ is the Euler constant defined in (48) of Report No.1), we can determine the constant B as

$$B = \log(2\pi) - 1 - \tfrac{1}{2}\log\pi - \gamma/2 = \tfrac{1}{2}\log(4\pi) - 1 - \gamma/2 = -0.0230957\ldots. \tag{28}$$

We can obtain another expression of B as follows (see Davenport [3] pp.81-82). The reflective property of $\xi(s)$ gives the identity

$$\frac{\xi'(s)}{\xi(s)} = -\frac{\xi'(1-s)}{\xi(1-s)}, \tag{29}$$

which, with (30) gives

$$B + \sum_n \left(\frac{1}{s-\rho_n} + \frac{1}{\rho_n} \right) = -B - \sum_n \left(\frac{1}{1-s-\rho_n} + \frac{1}{\rho_n} \right). \tag{30}$$

Thus,

$$\begin{aligned} B &= -\sum_n \frac{1}{\rho_n} - \frac{1}{2}\left(\sum_n \frac{1}{s-\rho_n} - \sum_n \frac{1}{s-(1-\rho_n)} \right) \\ &= -\sum_n \frac{1}{\rho_n} = -2\sum_{n=1}^{\infty} \frac{\sigma_n}{\sigma_n^2 + t_n^2}, \end{aligned} \tag{31}$$

[3]The Weierstrass factorization theorem asserts that a meromorphic function can be represented by a product of three factors: terms depending on the function's poles and zeros, and an associated non-zero *holomorphic* function.

Note that the sum terms in the parenthesis in the first line of the above equation cancel each other, because if ρ_n is a zero, so is $1-\rho_n$. To obtain the final expression in the above, we use the property that when $\rho_n = \sigma_n + it_n$ is a zero, so is its complex conjugate $\rho_n^* = \sigma_n - it_n$, thus we enumerate zeros in such a way that $\rho_n^* = \rho_{-n}$.

By substituting the above result $B = -\sum_n \frac{1}{\rho_n}$ into (30), we find

$$\frac{\xi'(s)}{\xi(s)} = \sum_n \frac{1}{s-\rho_n}, \tag{32}$$

and, thus

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_n \frac{1}{s-\rho_n} - \frac{g'(s)}{g(s)} = \sum_n \frac{1}{s-\rho_n} - \frac{1}{s-1} + \tfrac{1}{2}\log\pi - \tfrac{1}{2}\Psi\left(\tfrac{s}{2}+1\right). \tag{33}$$

2.3 Monotonicity Theorem for $|\xi(\mathbf{s})|$

By substituting (31) back into (23), we obtain

$$\begin{aligned}\xi(s) &= \tfrac{1}{2}\exp\left(-s\sum_n \frac{1}{\rho_n}\right)\prod_n\left(1-\frac{s}{\rho_n}\right)e^{s/\rho_n} = \tfrac{1}{2}\prod_n e^{-s/\rho_n}\left(1-\frac{s}{\rho_n}\right)e^{s/\rho_n}\\ &= \tfrac{1}{2}\prod_n\left(1-\frac{s}{\rho_n}\right).\end{aligned} \tag{34}$$

This is nothing but the product form

$$\xi(s) = \xi(0)\prod_n\left(1-\frac{s}{\rho_n}\right), \tag{35}$$

which Edwards attributes to Riemann (see [1] p. 18 and pp. 46-47). Taking the logarithm and differentiate, we obtain

$$\frac{\xi'(s)}{\xi(s)} = \sum_n \frac{1}{s-\rho_n}, \tag{36}$$

which is the same as (32). Taking the real part of this and using Lemma 2.1, we find

$$\frac{1}{|\xi(s)|}\frac{\partial|\xi(s)|}{\partial\sigma} = \Re\left(\sum_n \frac{1}{s-\rho_n}\right) = \sum_n \frac{\sigma-\sigma_n}{(\sigma-\sigma_n)^2+(t-t_n)^2}. \tag{37}$$

Thus, we arrive at the following theorem concerning the monotonicity of the $|\xi(s)|$ function, which Sondow and Dumitrescu [8] proved in a little more complicated way based on (23) instead of (34).

Theorem 2.1 (Monotonicity of Modulus Function $|\xi(s)|$)**.** *Let $\sigma_{\inf}$ be the infimum of the real parts of all zeros:*

$$\sigma_{\inf} = \inf_n\{\sigma_n\}.$$

Then the modulus $|\xi(\sigma+it)|$ is a monotone decreasing function of σ in the region $\sigma < \sigma_{\inf}$ for all real t. Likewise, the modulus is a monotone increasing function of σ in the region $\sigma_{\sup} < \sigma$, where

$$\sigma_{\sup} = \sup_n\{\sigma_n\} = 1 - \sigma_{\inf}.$$

Proof. The above discussion that has led to this theorem should suffice. □

Thus, if all zeta zeros are located on the critical line, the derivative of the modulus $|\xi(s)|$ is negative for $\sigma < \frac{1}{2}$, and positive for $\sigma > \frac{1}{2}$. Thus, we have shown the necessity of monotonicity of the modulus function $|\xi(s)|$, which has been one of major concerns towards a proof of the Riemann hypothesis.

Corollary 2.2 (Monotonicity of Modulus Function $|\xi(s)|$, if the Riemann hypothesis is true)**.** *If all zeta zeros are on the critical line, the modulus $|\xi(\sigma+it)|$ is a monotone decreasing function of σ in the left half plane, $\sigma < \frac{1}{2}$. Likewise, the modulus is a monotone increasing function of σ in the right half plane, $\sigma > \frac{1}{2}$.*

Proof. The above discussion that has led to this corollary should suffice. □

2.4 Functions $\mathbf{a}(\lambda, \mathbf{t}), \mathbf{b}(\lambda, \mathbf{t}), \alpha(\lambda, \mathbf{t}), \beta(\lambda, \mathbf{t})$ and their properties

By taking the imaginary part of both sides of (36) and evaluating them at $s = \frac{1}{2} + it$, and noting that $\xi(s)$ is real for $\sigma = \frac{1}{2}$, we obtain

$$\frac{1}{\xi(s)} \frac{\partial \Im(\xi(s))}{\partial \sigma}\bigg|_{s=\frac{1}{2}+it} = -\sum_n \frac{t-t_n}{(t-t_n)^2 + (\frac{1}{2}-\sigma_n)^2}, \tag{38}$$

Let us define real-valued functions $\Xi(t)$ and $b(t)$ by

$$\Xi(t) = \xi(\sigma+it)|_{\sigma=\frac{1}{2}}, \tag{39}$$

and

$$b(t) = \frac{\partial \Im\left\{\xi(\sigma+it)\right\}}{\partial \sigma}\bigg|_{\sigma=\frac{1}{2}} = -\frac{\partial \Re\{\xi(\sigma+it)\}}{\partial t}\bigg|_{\sigma=\frac{1}{2}} = -\Xi'(t). \tag{40}$$

Recall that $b(t)$ was informally introduced earlier in (10). Then, the LHS of (38) is equal to $\dfrac{b(t)}{\Xi(t)}$. Thus, $b(t)$ can be expressed as

$$b(t) = -\Xi(t) \cdot \sum_n \frac{t-t_n}{(t-t_n)^2 + (\frac{1}{2}-\sigma_n)^2}. \tag{41}$$

Differentiate (36) once more, and we obtain

$$\frac{\xi''(s)}{\xi(s)} - \left(\frac{\xi'(s)}{\xi(s)}\right)^2 = -\sum_n \frac{1}{(s-\rho_n)^2}. \tag{42}$$

By taking the real part of the above, and using Corollary 2.1, we have

$$-\frac{1}{|\xi(s)|}\frac{\partial^2|\xi(s)|}{\partial t^2} + \left(\frac{1}{|\xi(s)|}\frac{\partial|\xi(s)|}{\partial t}\right)^2 = \sum_n \frac{(t-t_n)^2 - \left(\frac{1}{2}-\sigma_n\right)^2}{\left(\frac{1}{2}+\sigma_n\right)^2}. \tag{43}$$

Noting that $\xi\left(\frac{1}{2}+it\right) = \Xi(t)$ is a real function, we have

$$\frac{|\Xi(t)|'}{|\Xi(t)|} = \frac{\Xi'(t)}{\Xi(t)}, \quad \text{and} \quad \frac{|\Xi(t)|''}{|\Xi(t)|} = \frac{\Xi''(t)}{\Xi(t)}. \tag{44}$$

By defining

$$a(t) = \tfrac{1}{2}\,\xi''(s)|_{s=\frac{1}{2}+it} = \frac{1}{2}\,\frac{\partial^2 \xi(\sigma+it)}{\partial \sigma^2}\bigg|_{\sigma=\frac{1}{2}} \tag{45}$$

evaluation of (42) at $s = \frac{1}{2} + it$ yields

$$\frac{2a(t)}{\Xi(t)} + \left(\frac{b(t)}{\Xi(t)}\right)^2 = \sum_n \frac{(t-t_n)^2 - (\frac{1}{2}-\sigma_n)^2}{\left[(t-t_n)^2 + (\frac{1}{2}-\sigma_n)^2\right]^2}. \tag{46}$$

In order to obtain the second term of the LHS in the last equation, we use Lemma 2.1:

$$\frac{1}{|\xi(s)|}\frac{\partial|\xi(s)|}{\partial t} = -\Im\left\{\frac{\xi'(s)}{\xi(s)}\right\}. \tag{47}$$

By writing

$$\begin{aligned}\xi'(s) = \frac{\partial\xi(s)}{\partial\sigma} &= \frac{\partial\Re\{\xi(s)\}}{\partial\sigma} + i\frac{\partial\Im\{\xi(s)\}}{\partial\sigma} \\ &= \frac{\partial\Im\{\xi(s)\}}{\partial t} + i\frac{\partial\Im\{\xi(s)\}}{\partial\sigma}.\end{aligned} \tag{48}$$

At $s = \frac{1}{2} + it$, the first term of RHS is zero and the second term is $ib(t)$. Thus (47) becomes at $s = \frac{1}{2} + it$

$$\left[\frac{1}{|\xi(s)|}\frac{\partial|\xi(s)|}{\partial t}\right]_{s=\frac{1}{2}+it} = -\frac{b(t)}{\Xi(t)}. \tag{49}$$

By dividing both sides of (41) by $\Xi(t)$, and differentiating both sides w.r.t. t, we obtain

$$\sum_n \frac{(t-t_n)^2 - (\frac{1}{2}-\sigma_n)^2}{[(t-t_n)^2]^2 + (\frac{1}{2}-\sigma_n)^2} = \frac{b'(t)\Xi(t) - b(t)\Xi'(t)}{\Xi(t)^2}, \tag{50}$$

Substituting this result into (46), we obtain a surprisingly simple result:

$$2a(t) = b'(t) = -\Xi''(t). \tag{51}$$

This final expression, however, could have been derived directly, by applying the Laplace equation to (45).

The above formulae, obtained for t on the critical line, can be generalized to any point $s = \frac{1}{2} + \lambda + it$:

Lemma 2.2. *Let us define*

$$2a(\lambda, t) = \frac{\partial^2\Re\left\{\xi\left(\frac{1}{2}+\lambda+it\right)\right\}}{\partial\lambda^2} = -\Re\left\{\xi''(\tfrac{1}{2}+\lambda+it)\right\}, \tag{52}$$

$$b(\lambda, t) = \frac{\partial\Im\left\{\xi\left(\frac{1}{2}+\lambda+it\right)\right\}}{\partial\lambda}. \tag{53}$$

Then, the following relations hold:

$$a(\lambda, t) = \tfrac{1}{2}b'(\lambda, t), \tag{54}$$

$$b(\lambda, t) = -\Re\left\{\xi'(\tfrac{1}{2}+\lambda+it)\right\} \tag{55}$$

Proof. By applying the Cauchy-Riemann equations and Laplace's equation, the above relations can be easily derived. □

We now derive similar functions and their relations by interchanging $\Re\left\{\xi\left(\frac{1}{2}+\lambda+it\right)\right\}$ and $\Im\left\{\xi\left(\frac{1}{2}+\lambda+it\right)\right\}$.

Corollary 2.3. *Let us define*

$$2\alpha(\lambda, t) = \frac{\partial^2\Im\left\{\xi\left(\frac{1}{2}+\lambda+it\right)\right\}}{\partial\lambda^2} = -\Im\left\{\xi''(\tfrac{1}{2}+\lambda+it)\right\}, \tag{56}$$

$$\beta(\lambda, t) = \frac{\partial\Re\left\{\xi\left(\frac{1}{2}+\lambda+it\right)\right\}}{\partial\lambda}. \tag{57}$$

Then the following relations hold:

$$\alpha(\lambda, t) = -\tfrac{1}{2}\beta'(\lambda, t), \tag{58}$$

$$\beta(\lambda, t) = \Im\left\{\xi'(\tfrac{1}{2}+\lambda+it)\right\}. \tag{59}$$

$$\frac{\partial a(\lambda, t)}{\partial\lambda} = \alpha'(\lambda, t), \qquad \frac{\partial\alpha(\lambda, t)}{\partial\lambda} = -a'(\lambda, t) \tag{60}$$

$$\frac{\partial\beta(\lambda, t)}{\partial\lambda} = b'(\lambda, t), \qquad \frac{\partial b(\lambda, t)}{\partial\lambda} = -\beta'(\lambda, t). \tag{61}$$

Proof. By applying the Cauchy-Riemann equations and Laplace's equation, the above relations can be easily derived. □

The functions $a(t)$ and other functions introduced in this section will be used in future reports.

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