No. 1: Euler's Zeta Function

Towards a Proof of the Riemann Hypothesis

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Abstract

In No. 1 of this series entitled "Towards a Proof of the Riemann Hypothesis," we briefly review the work by the greatest mathematician of the 18th century Leonhard Euler (1707-1783) on the zeta function $\zeta(k)$, which Bernhard Riemann (1826-1866) later generalized to the complex function $\zeta(s)$. We draw on heavily from Chapter 4 "Euler and His Legacy" in [3].

Key words: Euler's zeta function, harmonic series, Basel problem, Euler's product form.

1 Euler's zeta function

1.1 Harmonic series

From the of the 17th century to the first half of the 18th century, when such great mathematicians as Gottfried Wilhelm Leibniz (1646-1716), Isaac Newton (1642-1727), Jacob Bernoulli (1654-1705), Johann Bernoulli (1667-1748), and Daniel Bernoulli (1700-1782) appeared, was the period when differential and integral calculus was developed. Many mathematicians were interested in evaluating such series as

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \cdots, \quad (1)$$

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Figure 1: (a) Jacob Bernoulli (1654-1705), (b) Johann Bernoulli (1667-1748), (c) Daniel Bernoulli (1700-1782), Sources: Wikipedia

where k was an integer. The case $k = 1$, i.e.,

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \cdots . \tag{2}$$

is known as the *harmonic series*. The name derives from concept of tones in music [6]. The fact that the above series diverges, i.e., the sum is infinite, was first apparently proved by a French monk Nicole Orseme (c. 1320-1325 till 1382) around 1350. His proof, however, was lost, and it was not until the 17th century when the Italian mathematician Pietro Mengoli (1626-1686) published a proof. Since then numerous proofs have been published The following proof given in many articles is credited to the aforementioned Orseme. We rearrange (2) and proves its divergence, as shown below.

$$\begin{aligned}
\zeta(1) &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots + \frac{1}{16}\right) + \cdots \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty.
\end{aligned} \tag{3}$$

Another simple proof is to replace each $\frac{1}{2k-1}$ by $\frac{1}{2k}$ in (2). If we assume $\zeta(1)$ is finite, we should be able to establish the following inequality:

$$\begin{aligned}
\zeta(1) &> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} + \frac{1}{6} + \frac{1}{8} + \frac{1}{8} + \cdots \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \\
&= \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \frac{1}{2} + \xi(1).
\end{aligned} \tag{4}$$

But this cannot be true, if $\zeta(1)$ is finite. Hence, we have proved that $\zeta(1)$ must be ∞ by the method of contradiction.

1.2 The Basel Problem

The problem of evaluating $\zeta(2)$, which was posed in 1644 by the aforementioned Pietro Mengoli, has been referred to as the "Basel problem"[1].

It was not so difficult to see that $\zeta(2)$ must be finite. This can be shown, for instance, by the following manipulation of the series

$$\begin{aligned}
\zeta(2) &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} + \cdots \\
&< 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(n-1)n} + \cdots \\
&< 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) + \cdots \\
&< 2 - \frac{1}{n+1} + \cdots .
\end{aligned} \tag{5}$$

By letting n go to infinity, one can readily find that RHS of (5), which is an upper bound of $\zeta(2)$, converges to 2. The German mathematician Christian Goldbach (1690-1764) showed in his letter dated January 31, 1729 to Daniel Bernoulli (1700-1782), the second son of Johann Bernoulli, that $\zeta(2)$ must be bounded by

$$1 + \frac{16223}{25200} < \zeta(2) < 1 + \frac{30197}{46800}, \quad \textit{i.e.}, \quad 1.6437 < \zeta(2) < 1.6453.$$

Figure 2: Leonhard Euler (1707-1783), Source: Wikipedia

Another Swiss mathematician Leonhard Euler (1707-1783)[5], who was then a professor of the Russian at St. Petersburg Academy together with Daniel Bernoulli, came up with an ingenious way to derive an approximation $\zeta(2) \approx 1.644934$ and published it in 1732. His derivation used differentiation and integration in a very skillful manner. Euler started with the identity

$$\frac{d\log(1-x)}{dx} = -\frac{1}{1-x} = -(1+x+x^2+x^3+\cdots), \quad |x|<1, \tag{6}$$

which by integration becomes

$$\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right), \ |x|<1. \tag{7}$$

By dividing both sides by x and integrating once more, we find

$$\int_0^u \frac{\log(1-x)}{x}\,dx = -\left(u + \frac{u^2}{2^2} + \frac{u^3}{3^2} + \frac{u^4}{4^2} + \cdots\right), \quad |u|<1. \tag{8}$$

We then change the variable to $y = 1-x$, obtaining

$$\int \frac{\log(1-x)}{x}\,dx = -\int \frac{\log y}{1-y}\,dy = -\int \left(\sum_{n=0}^{\infty} y^n\right) \log y. \tag{9}$$

By using integration by parts, we have

$$\int y^n \log y\,dy = \frac{y^{n+1}\log y}{n-1} - \frac{y^{n+1}}{(n+1)^2} + \text{const.} \tag{10}$$

Thus,

$$\int \frac{\log(1-x)}{x}\,dx = \sum_{n=0}^{\infty}\left[\frac{(1-x)^{n+1}}{(n+1)^2} - \frac{(1-x)^{n+1}\log(1-x)}{n+1}\right] + \text{const.} \tag{11}$$

[1]Although the problem was posed by Mengoli, a professor at the University of Bologna, the name "the Basel problem" came from the publishing location, Basel, of the book *Tractatus de seriebus infinitis* of Jacob Bernoulli (1654-1703), published posthumously in 1713 by his nephew Nicolaus Bernollui (1587-1759), which brought the problem to the attention of a wide audience.

Then he obtained the following result for the definite integral:

$$\int_0^u \frac{\log(1-x)}{x}\,dx = \sum_{n=0}^{\infty}\left[\frac{(1-u)^{n+1}}{(n+1)^2} - \frac{(1-u)^{n+1}\log(1-u)}{n+1}\right] - \xi(2), \tag{12}$$

from which, he found

$$\zeta(2) = \sum_{n=0}^{\infty}\frac{u^{n+1}+(1-u)^{n+1}}{(n+1)^2} + \log u \cdot \log(1-u), \tag{13}$$

where the following identity was used in obtaining the last term:

$$-\sum_{n=0}^{\infty}\frac{(1-u)^{n+1}}{n+1} = \log u, \tag{14}$$

which readily follows by setting $1-x=u$ in (7). By setting $u=\frac{1}{2}$ in (13), the following expression follows.

$$\zeta(2) = \sum_{n=0}^{\infty}\frac{1}{2^n(n+1)^2} + (\log 2)^2. \tag{15}$$

Because of the term 2^n in the denominator of RHS, the above series converges fast. So after summing only the first 11 terms, one can obtain

$$\sum_{n=0}^{\infty}\frac{1}{2^n(n+1)^2} \approx 1.164481. \tag{16}$$

In order to obtain an approximate value of $\log 2$, the following identity, obtainable from (7) is used:

$$\begin{aligned}\log\frac{1+t}{1-t} &= \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots\right) - \left(-t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \cdots\right)\\ &= 2\left(t + \frac{t^3}{3} + \frac{t^5}{5} + \cdots\right).\end{aligned}$$

By setting $t=\frac{1}{3}$, one can find by summing the first 9 terms

$$\log 2 \approx 0.693147, \quad \text{or} \quad (\log 2)^2 \approx 0.480453.$$

Thus, Euler obtained

$$\zeta(2) \approx 1.644934. \tag{17}$$

Euler subsequently developed what is now known as the Euler MacLaurin summation method, and obtained

$$\zeta(2) \approx 1.64493406684822643647.$$

1.3 Discovery of $\zeta(2) = \frac{\pi^2}{6}$

But the above results were both numerical approximate results, with no insight to the meaning of the value 1.64493.... But several years later, in 1735, Euler discovered the following closed form expression for $\zeta(2)$, which surprised Euler himself.

$$\zeta(2) = \frac{\pi^2}{6}. \tag{18}$$

Euler's original proof was as follows. Consider the Taylor series expansion

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots. \tag{19}$$

Dividing the above by x,

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots. \tag{20}$$

He then used the following factorization theorem. Consider a polynomial function $f(x)$ of degree n, whose constant term is 1, i.e., $f(0) = 1$. If the roots of $f(x) = 0$ are $\alpha_i,\ \ 1 \le i \le n$, then the following factorization must hold

$$f(x) = \prod_{i=1}^{n} \left(1 - \frac{x}{\alpha_i}\right). \tag{21}$$

This factorization holds even for $n = \infty$ under certain conditions, and this is known as Weierstrass' factorization theorem, owing to the Germann mathematician Karl Weierstrass (1815-1897), which we will discuss in a later section. Needless to say, this theorem did not exist at the time of Euler. Euler assumed the above factorization was applicable to $\frac{\sin x}{x}$, and noting that $x = \pm k\pi,\ \ k \ge 1$ are roots of the function $\frac{\sin x}{x}$, he obtained the following expression

$$\begin{aligned}\frac{\sin x}{x} &= \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right)\cdots \\ &= \prod_{k=1}^{\infty}\left(1 - \frac{x^2}{k^2\pi^2}\right). \end{aligned} \tag{22}$$

By comparing the coefficients of the x^2 terms in (20) and RHS of (20), he found

$$\sum_{k=1}^{\infty} \frac{1}{k^2\pi^2} = \frac{1}{3!}, \tag{23}$$

from which (18) readily follows.

1.4 Euler's Second Derivation

Euler, not satisfied in this proof based on the conjecture that the product form (21) should be valid for the infinite polynomial function (20), continued working on the problem and came up with a more rigorous derivation of (18) eight years later, in 1743, using the integration

$$\int_0^1 \frac{1}{\sqrt{1-x^2}} \sin^{-1} x \, dx. \tag{24}$$

He began with the well known formula

$$\frac{d \sin^{-1} x}{dx} = \frac{1}{\sqrt{1-x^2}}, \tag{25}$$

which gave rise to

$$\frac{1}{2}\frac{d(\sin^{-1} x)^2}{dx} = \frac{\sin^{-1} x}{\sqrt{1-x^2}}, \tag{26}$$

integration of which led to

$$\int_0^1 \frac{\sin^{-1} x}{\sqrt{1-x^2}} = \frac{1}{2}\left.(\sin^{-1} x)^2\right|_{x=0}^{1} = \frac{\pi^2}{8}. \tag{27}$$

He then considered the Taylor series expansion of $(1-u)^{-\frac{1}{2}}$:

$$\frac{1}{\sqrt{1-u}} = \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)!}{2^n n!} u^n. \tag{28}$$

By setting $u = x^2$ in the above and substituting the result into RHS of (25), he obtained

$$\frac{d\sin^{-1} x}{dx} = \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)!}{2^n n!} x^{2n}, \tag{29}$$

integration of which yields

$$\sin^{-1} x = \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)!}{2^n n!(2n+1)} x^{2n+1}. \tag{30}$$

By substituting this into (24) he obtained

$$\int_0^1 \frac{1}{\sqrt{1-x^2}} \sin^{-1} x \, dx = \int_0^1 \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)! x^{2n+1}}{2^n n!(2n+1)\sqrt{1-x^2}} \, dx. \tag{31}$$

In order to evaluate the integral

$$\int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}} \, dx$$

he set $x = \sin t$, obtaining

$$\int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}} \, dx = \int_0^{\frac{\pi}{2}} \sin^{2n+1} t \, dt. \tag{32}$$

By writing the integral on RHS as I_{2n+1}, Euler obtained by integration by parts

$$\begin{aligned} I_{2n+1} &= -\left[\cos t \cdot \sin^{2n} t\right]_{x=0}^{\pi/2} + \int_0^{\pi/2} 2n \cos^2 t \sin^{2n-1} t \, dt \\ &= 2n \int_0^{\pi/2} (1 - \sin^2 t) \sin^{2n-1} t \, dt = 2n I_{2n-1} - 2n I_{2n+1}. \end{aligned} \tag{33}$$

Thus,

$$(2n+1) I_{2n+1} = 2n I_{2n}, \tag{34}$$

By recursion, he obtained

$$I_{2n+1} = \frac{2n \cdot 2(n-1) \cdots 2}{(2n+1)(2n-1) \cdots 3 \cdot 1} I_1. \tag{35}$$

But

$$I_1 = \int_0^1 \sin t \, dt = 1,$$

from which he found

$$I_{2n+1} = \frac{2^n n!}{(2n+1)(2n-1) \cdots 3 \cdot 1}. \tag{36}$$

By changing the order of integration and summation in (31), and substituting the above, Euler obtained

$$\int_0^1 \frac{1}{\sqrt{1-x^2}} \sin^{-1} x \, dx = \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdots (2n-1) I_{2n+1}}{2^n n!(2n+1)} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}. \tag{37}$$

By equating this to (27), he obtained

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}. \tag{38}$$

By writing $\zeta(2)$ as

$$\zeta(2) = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \cdots \right)$$
$$= \frac{\pi^2}{8} + \frac{1}{4}\xi(2), \tag{39}$$

from which (18) ensues.

1.5 Evaluation of $\zeta(\mathbf{2k})$

By comparing the coefficients of the $x^4, x^6, x^8, \cdots$ terms in (20) and RHS of (20), Euler obtained

$$\zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \quad \zeta(8) = \frac{\pi^8}{9450}, \quad \zeta(10) = \frac{\pi^{10}}{93555},$$
$$\zeta(12) = \frac{691\pi^{12}}{638512875}, \cdots, \zeta(20) = \frac{174611\pi^{20}}{1531329465290625}. \tag{40}$$

Euler obtained a general formula $\zeta(2k)$ in terms of the *Bernoulli numbers*[2]:

$$\zeta(2k) = \sum_{n=0}^{\infty} \frac{1}{n^{2k}} = (-1)^{k-1} \frac{B_{2k}}{2(2k)!} (2\pi)^{2k}. \tag{42}$$

Alternatively, the Bernoulli numbers B_k can be defined as the coefficients of the Taylor expansion of $\dfrac{te^t}{e^t - 1}$:

$$\frac{te^t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}. \tag{43}$$

For $k \geq 1$, $(-1)^{k-1} B_{2k} > 0$. The first several Bernoulli numbers are

$$B_1 = \tfrac{1}{2}, \quad B_2 = \tfrac{1}{6}, \quad B_4 = -\tfrac{1}{30}, \quad B_6 = \tfrac{1}{42}, \quad B_8 = -\tfrac{1}{30}, \quad B_{10} = \tfrac{5}{66}, \quad B_{12} = -\tfrac{691}{2730},$$
$$B_{2k+1} = 0, \quad (k \geq 1). \tag{44}$$

In the definition (43), the function $\dfrac{te^t}{e^t - 1}$ is called the *generating function* of the Bernoulli numbers. Another definition of the Bernoulli numbers is the following recursion formula:

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k = n + 1, \quad n = 0, 1, 2, \ldots. \tag{45}$$

1.6 Evaluation of $\zeta(\mathbf{2k+1})$

For any odd number $n = 2k + 1$, however, Euler could not find a closed form expression for the value $\zeta(n)$, and Euler eventually gave up. From (42) we know that $\zeta(n)$ is an irrational number for an even argument $n = 2k$, since since it is a product of a rational number given in terms of a Bernoulli number and an irrational number π^n. For n odd, however, no such representation was given, so it was not known for a long time that $\zeta(n)$ with an odd argument was whether $\zeta(n)$ is rational or not, let alone transcendental[3] or not. But in

[2]The Bernoulli numbers B_k were originally introduced by Jacob Bernoulli in order to express the sum of powers:

$$1^k + 2^k + 3^k + \cdots n^k = \sum_{j=0}^{k} \binom{k}{j} B_j \frac{n^{k+1-j}}{k+1-j}. \tag{41}$$

[3]A transcendental number is a real or complex number that is not algebraic, i.e., it is not a root of a non-zero polynomial equation with rational coefficients. The best-known transcendental numbers are π and e.

1978 the Greek-French mathematician Roger Apéry (1916-1994) proved that $\zeta(3)$ is an irrational number, which is now known as the Apéry's theorem. In 2000 Rivoal Tanguy reported that there are infinitely many irrational $\zeta(n)$ at odd integers. In 2001, the Russian number theorist Wadim Zudlin showed that one of the numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ must be irrational. Other than these limited results, any useful representation of $\zeta(n)$ for odd integers n has remained unsolved since Euler gave it up around 1745, 270 years ago!

2 Euler Product and Distribution Density of Prime Numbers

2.1 Euler Constant

In the previous section we discussed the harmonic series diverges. In order to see its divergent speed, i.e, how fast or slow the sum grow without bound, we compare it with the integration of $\frac{1}{x}$. By plotting the curve $y = \frac{1}{x}, \quad 0 \leq x \leq n$, we readily find the inequality

$$\int_1^{n+1} \frac{1}{x}\, dx < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < 1 + \int_1^n \frac{1}{x}\, dx.$$

In other words,

$$\log(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < 1 + \log n, \tag{46}$$

from which we readily see

$$0 < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log(n+1) < 1 + \log n - \log(n+1) < 1. \tag{47}$$

The series $\sum_{k=1}^{n} \frac{1}{k} - \log(n+1)$ increases monotonically as n goes to infinity, but it is bounded from above by unity. Therefore the limit

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log(n+1) \right) \tag{48}$$

must exist. The number γ is called the *Euler constant* (also called Euler-Mascheroni[4]). In order to evaluate its value, consider the Taylor expansion of $\log(1+x)$:

$$\log(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots . \tag{49}$$

By setting $x = \frac{1}{k}$ in the above, we have

$$\log \frac{k+1}{k} = \frac{1}{k} - \frac{1}{2k^2} + \frac{1}{3k^3} - \frac{1}{4k^4} + \cdots , \tag{50}$$

from which we find

$$\frac{1}{k} - \log(k+1) + \log k = \frac{1}{2k^2} - \frac{1}{3k^3} + \frac{1}{4k^4} - \cdots . \tag{51}$$

By setting $k = 1, 2, 3, \ldots, n$ in the above and summing both sides of the resulting equations, we obtain

$$\sum_{k=1}^{n} \frac{1}{k} - \log(n+1) = \tfrac{1}{2} \sum_{k=1}^{n} \frac{1}{k^2} - \tfrac{1}{3} \sum_{k=1}^{n} \frac{1}{k^3} + \tfrac{1}{4} \sum +k = 1^n \frac{1}{k^4} - \cdots , \tag{52}$$

from which we can evaluate the Euler constant:

$$\gamma \approx 0.5772218 \tag{53}$$

Note that if we let $n \to \infty$ in (52), we find

$$\gamma = \tfrac{1}{2}\zeta(2) - \tfrac{1}{3}\zeta(3) + \tfrac{1}{4}\zeta(4) - \cdots . \tag{54}$$

[4]Lorenzo Moscheroni (1750-1800) was an Italian mathematician constant. In 1790 he calculated Euler's constant to 32 digits, although it was found later that only the first 19 digits were correct. Despite this error, γ is called the Euler-Moscheroni constant.

2.2 The Harmonic Series as Infinite Product

Euler found that the harmonic series can be expresses as an infinite product:

$$\sum_{n=1}^{\infty} \frac{1}{n} = \prod_p \frac{1}{1-\frac{1}{p}}, \tag{55}$$

where the infinite product in RHS is taken over all primes p. In fact, Euler wrote RHS in the form

$$\prod_p \frac{p}{p-1} = \frac{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdots}{1 \cdot 2 \cdot 4 \cdot 6 \cdot 10 \cdot 12 \cdots}. \tag{56}$$

Since the harmonic series diverges, and thus the expression (55) is not of much use, it is important to recognize that the expression shows uniqueness in the prime factorization. First, we write

$$\frac{1}{1-\frac{1}{p}} = 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots. \tag{57}$$

By setting $p = 2, 3, 5, \ldots$, we obtain

$$\begin{aligned}
\frac{1}{1-\frac{1}{2}} &= 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots, \\
\frac{1}{1-\frac{1}{3}} &= 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots, \\
\frac{1}{1-\frac{1}{5}} &= 1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots, \\
\cdots
\end{aligned}$$

Thus, we have

$$\left(\frac{1}{1-\frac{1}{2}}\right)\left(\frac{1}{1-\frac{1}{3}}\right) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{2 \cdot 3} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{2^2 \cdot 3} + \frac{1}{3^2 \cdot 2} + \frac{1}{3^3} + \cdots. \tag{58}$$

Note that in RHS all the natural numbers that are product of primes 2 and 3 appear and only once. Similarly, we expand

$$\left(\frac{1}{1-\frac{1}{2}}\right)\left(\frac{1}{1-\frac{1}{3}}\right)\left(\frac{1}{1-\frac{1}{5}}\right) = \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right)\left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots\right)\left(1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots\right) \tag{59}$$

then the denominators of terms in RHS all the numbers that can be factored into the primes 2, 3, and 5 appear and only once. By repeating this argument, we see that when we expand $\prod_p \left(1 - \frac{1}{p}\right)$, each natural number appears in the denominator once. Therefore, this infinite product should be equal to LHS of (55).

2.3 Euler Product for $\zeta(\sigma)$

The product form (55) can be generalized to

$$\zeta(\sigma) = \sum_{n=1}^{\infty} \frac{1}{n^\sigma} = \prod_p \frac{1}{1-\frac{1}{p^\sigma}}. \tag{60}$$

where σ is any real number. Obviously the product factorization (55) is a special case $\sigma = 1$. In order to prove the above, let us define $P(k)$ for a natural number k:

$$P(k) = \prod_{p \le k} \frac{1}{1-\frac{1}{p^\sigma}}, \tag{61}$$

in which the product is taken for primes p that do not exceed k. The product

$$P(k) = \prod_{p \le k} \left(1 + \frac{1}{p^{\sigma}} + \frac{1}{p^{2\sigma}} + \frac{1}{p^{3\sigma}} + \cdots \right), \tag{62}$$

can be expanded into the form

$$P(k) = \sum \frac{1}{n^{\sigma}}, \tag{63}$$

where n are natural numbers that do not have as their factors any primes that are larger than k, and all such natural numbers appear in the summation of RGS of the above expression only once.

The difference between $\zeta(\sigma)$ of (60) and $P(k)$ can be expressed as

$$\zeta(\sigma) - P(k) = \sum \frac{1}{m^{\sigma}}, \tag{64}$$

where m are the natural numbers that did not appear in the sum expression (63). In other words, they are the natural numbers that contain, as their factor, at least one prime that is grater than k. Such natural numbers are obviously greater than k, thus we have

$$\zeta(\sigma) - P(k) \le \sum_{n=k+1}^{\infty} \frac{1}{n^{\sigma}}. \tag{65}$$

For $\sigma > 1$, the series $\zeta(\sigma) = \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}}$ converges, thus

$$\lim_{k \to \infty} \sum_{n=k+1}^{\infty} \frac{1}{n^{\sigma}} = 0, \tag{66}$$

that is to say

$$\zeta(\sigma) = \lim_{k \to \infty} P(k) = \prod_{p} \left(\frac{1}{1 - \frac{1}{p^{\sigma}}} \right). \tag{67}$$

Thus, we have proved (60) for any real number σ.

Earlier, we compared the harmonic series with an integral of the function $\frac{1}{x}$. We now compare $\zeta(\sigma)$ with the function $\frac{1}{x^{\sigma}}$, and can show

$$\zeta(\sigma) = \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} < \int_{1}^{\infty} \frac{1}{x^{\sigma}}\, dx = \frac{1}{\sigma - 1}, \tag{68}$$

thus $\zeta(\sigma)$ converges for $\sigma > 1$.

2.4 Frequency of Primes

Let us return to the case $\sigma = 1$. Then (61) and (63) become

$$P(k) = \prod_{p \le k} \frac{1}{1 - \frac{1}{p}} = \sum \frac{1}{n}, \tag{69}$$

where n are such natural numbers, as defined in (63), that cannot be divided by any prime larger than k. Obviously any natural number $n \le k$ possesses such property, thus we can establish the following inequalities

$$P(k) \ge \sum_{n \le k} \frac{1}{n} > \int_{1}^{k} \frac{dt}{t} = \log k. \tag{70}$$

Let us consider the Taylor expansion

$$\log(1-t) = -\left(t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4} + \cdots\right), \tag{71}$$

and set $t = \frac{1}{p}$, obtaining

$$\log \frac{1}{1-\frac{1}{p}} = -\log\left(1 - \frac{1}{p}\right) = \frac{1}{p} + \frac{1}{2p^2} + \frac{1}{3p^3} + \frac{1}{4p^4} + \cdots . \tag{72}$$

Thus, we obtain from (70) and (71)

$$\log P(k) = \sum_{p\leq k}\left(\frac{1}{p} + \frac{1}{2p^2} + \frac{1}{3p^3} + \frac{1}{4p^4} + \cdots\right) > \log\log k. \tag{73}$$

The summed terms in the middle expression above, excluding $\dfrac{1}{p}$ can be bounded as follows:

$$\frac{1}{2p^2} + \frac{1}{3p^3} + \frac{1}{4p^4} + \cdots < \frac{1}{2p^2}\left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \frac{1}{2p^2\left(1-\frac{1}{p}\right)} = \frac{1}{2}\left(\frac{1}{p-1} - \frac{1}{p}\right). \tag{74}$$

By summing the above over all primes not larger than k, we have

$$\sum_{p\leq k}\left(\frac{1}{2p^2} + \frac{1}{3p^3} + \frac{1}{4p^4} + \cdots\right) < \frac{1}{2}\sum_{p\leq k}\left(\frac{1}{p-1} - \frac{1}{p}\right) < \frac{1}{2}\sum_{n\leq k}\left(\frac{1}{n-1} - \frac{1}{n}\right) = \frac{1}{2}\left(1 - \frac{1}{k}\right) < \frac{1}{2}. \tag{75}$$

By substituting this result into (73), we find

$$\sum_{p\leq k}\frac{1}{p} + \frac{1}{2} > \log\log k. \tag{76}$$

Thus, by letting $k \to \infty$, we obtain

$$\sum_{p}\frac{1}{p} = \infty. \tag{77}$$

This result implies not only that there are infinitely many primes, but also the frequency with which prime numbers appear is large to the extent that $\sum \dfrac{1}{p}$ diverges. As a comparison, consider a set of all numbers that are powers of 2, i.e., $\{2, 2^2, 2^3, \cdots\}$. Obviously, the elements of this set are also countably infinite, but the sum of their inverses

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = 2,$$

which means that powers of 2 appear less frequently than primes.

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