PRINCETON UNIVERSITY
Department of Electrical Engineering
ELE 525: Random Processes in Information Systems
Final Examination
January 20 (Mon), 2014; 1:00 pm-4:00 pm
This is a **Closed Book Exam**.

Problem 1: Some definitions (20 points)
Give a brief definition of each term below

(a) A semi-Markov Process and a continuous-time Markov chain (CTMC)

(b) Karhunen-Loéve expansion

Problem 2: Brownian motion (20 points)

Let $W(t)$ be Brownian motion (a.k.a. the Wiener process) with $W(0) = 0$ and $\mathrm{Var}[W(t)] = \alpha t$. In answering the following questions, you may refer to the five properties of the process $W(t)$: 1. Spatial homogeneity, 2. Temporal homogeneity, 3. Independent increments, 4. Markov property, and 5.Gaussian property.

(a) Show the following properties of $W(t)$

$$R_W(t,s) = \alpha \min\{s,t\}, \quad s,t > 0. \tag{1}$$

$$\mathrm{Var}[W(t) - W(s)] = \alpha|t-s|, \quad s,t > 0. \tag{2}$$

(b) Define a random process $Y(t)$ by

$$Y(t) = e^{W(t)}.$$

Find its mean $E[Y(t)$ and variance $\mathrm{Var}[Y(t)]$.

Problem 3: Estimation based on past values (20 points) Consider estimating a time continuous process $X(t)$ in terms of two preceding values, $X(t-\tau_1)$ and $X(t-\tau_2)$, where $\tau_2 \geq \tau_1 \geq 0$, using the following linear estimation scheme:

$$\hat{X}(t) = \beta_1 X(t-\tau_1) + \beta_2 X(t-\tau_2). \tag{3}$$

Assume that $X(t)$ is a real-values WSS (wide-sense stationary) process.

(a) Find β_1 and β_2 such that the mean-square error (MSE) is minimized.

(b) Suppose that given $X(t-\tau_1)$, the knowledge of the older value $X(t-\tau_2)$ does not improve the estimation $\hat{X}(t)$. Show that the autocorrelation function must take the following form:

$$R_X(\tau) = R_X(0)e^{-\alpha|\tau|}.$$

Problem 4: A hidden Markov model (HMM) and estimation algorithms (40 points)
Consider an information sequence $\{I_t\}$ $I_t \in \{+1, -1\}$, $t \in \mathcal{T} = \{0, 1, \ldots, T\}$. Assume the I_t's are i.i.d. with $P[I_t = +1] = P[I_t = -1] = 1/2$ for all $t \in \mathcal{T}$.

The information sequence is sent over a linear and time-invariant channel, which is dispersive so that a "+1" signal sent at time t appears at the channel output at times $t, t+1, \ldots, t+d$ with amplitudes $h_0, h_1, \ldots, h_d$, respectively. Thus, the channel output at time t is given by

$$O_t = \sum_{i=0}^{d} I_{t-i} h_i, \quad t \in \mathcal{T} \tag{4}$$

The observation sequence Y_t is given by

$$Y_t = O_t + N_t, \quad t \in \mathcal{T} \tag{5}$$

where the noise $\{N_t\}$ are i.i.d. Gaussian variables with zero mean and variance σ^2.

Consider the case $d = 1$, hence the parameters of interest are $\boldsymbol{\theta} = (h_0, h_1, \sigma)$. In the questions (a) through (e), assume that the parameters $\boldsymbol{\theta}$ are fixed and known.

(a) *Formulation of a hidden Markov model*
Formulate the system in terms of a hidden Markov model (HMM)[1]

Define a set of Markov states, denoted $\mathcal{S}$, and draw the state transition diagram.

(b) *Conditional joint probabilities* $p(j, y|i)$

Find the following conditional joint probabilities for all state pairs $i, j \in \mathcal{S}$, and $y \in \mathcal{R}$.

$$p(j, y|i)\, dy = P[S_t = j, y < Y_t \leq y + dy | S_{t-1} = i], \quad i, j \in \mathcal{S}, -\infty < y < \infty. \tag{6}$$

(c) *Posterior probability and joint probability*

When the observation sequence $\boldsymbol{Y}_0^T = \boldsymbol{y}$ is given, the posterior probability of the state sequence $\boldsymbol{S}_0^T = \boldsymbol{s}$ is $\pi(\boldsymbol{s}|\boldsymbol{y}) = \frac{p(\boldsymbol{s},\boldsymbol{y})}{p(\boldsymbol{y})}$, and the initial probability is denoted by $\pi(s_0, y_0)$, i.e.,

$$\pi(s_0, y_0) dy = P[S_0 = s_0, y_0 < Y_0 \leq y_0 + dy].$$

Express the joint probability $p(\boldsymbol{s}, \boldsymbol{y})$ in terms of $\pi(s_0, y_0)$ and the conditional joint probabilities defined in part (b).

[1] *Definition (Hidden Markov Model).* A Markov process (S_t, Y_t) is called a partially observable Markov process or HMM, if its state transition probability does not depend on $Y_{t-1} = y'$, i.e.,

$$p_{S_t, Y_t | S_{t-1}, Y_{t-1}}(j, y | i, y') = p(j, y|i).$$

(d) *Auxiliary variables and a recursion formula*

Consider the following auxiliary variables:

$$\alpha_t(j, \boldsymbol{y}_0^t) = \max_{\boldsymbol{s}_0^{t-1}} P[\boldsymbol{S}_0^{t-1} = \boldsymbol{s}_0^{t-1}, S_t = j. \boldsymbol{Y}_0^t = \boldsymbol{y}_0^t] \tag{7}$$

Find the recursion formula for the auxiliary variables.

(e) *Simplify the algorithm for a MAP sequence estimation*

Simplify the above recursion formula and describe the resultant algorithm to find a MAP (maximum a posteriori probability) state sequence estimate $\hat{\boldsymbol{s}}^*$ from the observation $\boldsymbol{Y}_0^T = \boldsymbol{y}$. Then find the MAP information sequence $\hat{\boldsymbol{i}}^* = (\hat{i}_0^*, \hat{i}_1^*, \ldots, \hat{i}_T^*)$.

(f) *A maximum likelihood estimate (MLE) of the channel parameters*

Now assume that the information sequence $\boldsymbol{i}_0^T$ sent over the channel is known to the receiver, but now the channel parameters $\boldsymbol{\theta}$ are unknown and have to be estimated. Obtain a maximum likelihood estimate (MLE) of the parameters $\boldsymbol{\theta}$ from the observation $\boldsymbol{y}$ and the given information sequence $\boldsymbol{i}$.