# Lecture 6:
# Rayleigh, Rice and Lognormal Distributions
# Transform Methods and the Central Limit Theorem

**ELE 525: Random Processes in Information Systems**

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Textbook: Hisashi Kobayashi, Brian L. Mark and William Turin, ***Probability, Random Processes and Statistical Analysis*** (Cambridge University Press, 2012)

## ❖ Propagation in a radio channel

Let $L$ (>1) be the loss or attenuation factor.

Divide the path between the transmitter and receiver into contiguous and disjoint segments. The overall loss $L$ is the product of the loss within each segment:

$$L = \prod_{i=1}^{n} L_i \tag{7.54}$$

$$Y = \sum_{i=1}^{n} Y_i, \tag{7.55}$$

where we set

$$Y = \ln L, \text{ and } Y_i = \ln L_i, \text{ for } i = 1, 2, \ldots, n.$$

From the **central limit theorem** (CLT), we see that $Y$ is asymptotically normally distributed.

Therefore, the overall attenuation factor is log-normally distributed.

## ❖ decibel (dB) representation

$$Z = 10\log_{10} L = 10\frac{\ln L}{\ln 10} = (10\log_{10} e)Y \quad [\mathrm{dB}] \tag{7.56}$$

$$\begin{aligned} E[Z] &= \frac{10}{\ln 10}\mu_Y = \frac{10}{\ln 10}\left[\ln\mu_L - \frac{1}{2}\ln\left(1+\frac{\sigma_L^2}{\mu_L^2}\right)\right] \\ &= 10\left[\log_{10}\mu_L - \frac{1}{2}\log_{10}\left(1+\frac{\sigma_L^2}{\mu_L^2}\right)\right] \quad [\mathrm{dB}]. \end{aligned} \tag{7.57}$$

$$\sigma_Z = \frac{10}{\ln 10}\sigma_Y = 10\sqrt{\log_{10} e \log_{10}\left(1+\frac{\sigma_L^2}{\mu_L^2}\right)} \quad [\mathrm{dB}]. \tag{7.58}$$

# 7.5 Rayleigh and Rice distributions

## 7.5.1 Rayleigh distribution

Let $X$ and $Y$ be independent RVs with $N(0, \sigma^2)$. We define

$$R = \sqrt{X^2 + Y^2}, \quad R \geq 0. \tag{7.59}$$

Then its PDF is

$$f_R(r) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}, \quad r \geq 0. \tag{7.60}$$

called the Rayleigh distribution.

❖ Derivation of (7.60):
Writing $X=\sigma U_1$ and $Y=\sigma U_2$, where $U_1$ and $U_2$ are from $N(0, 1)$ , we see

$$R^2 = \sigma^2 \chi_2^2$$

By setting n=2 in (7.2) we find

$$f_{\chi_2^2}(x) = \frac{e^{-\frac{x}{2}}}{2\Gamma(1)} = \frac{1}{2} e^{-\frac{x}{2}}, \quad x \geq 0. \tag{7.61}$$

Set

$r^2 = \sigma^2 \nu$ ($r$ and $\nu$ are the values that the RVs $R$ and $\chi^2$ take, respectively).

$2r\,dr = \sigma^2\,d\nu$.

Then equating $f_{\chi_2^2}(\nu)\,d\nu = f_R(r)\,dr$, we readily obtain (7.60).

❖ Alternative derivation of (7.60):

$$X = R\cos\Theta, \quad \text{and } Y = R\sin\Theta, \quad R \leq 0, \quad \Theta \in [0, 2\pi]. \tag{7.63}$$

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma^2}\exp\left\{-\frac{x^2+y^2}{2\sigma^2}\right\} = \frac{1}{2\pi\sigma^2}\exp\left\{-\frac{r^2}{2\sigma^2}\right\} \tag{7.64}$$

$$\left|J\left(\frac{x,y}{r,\theta}\right)\right| = \left|\det\begin{bmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{bmatrix}\right| = \left|\det\begin{bmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{bmatrix}\right| = r, \tag{7.65}$$

$$f_{R\Theta}(r,\theta) = J\left(\frac{x,y}{r,\theta}\right)f_{X,Y}(x,y) = \frac{1}{2\pi}\cdot\frac{r}{\sigma^2}\exp\left\{-\frac{r^2}{2\sigma^2}\right\} = f_\Theta(\theta)f_R(r), \tag{7.66}$$

Hence, we obtain (7.60) and

$$f_\Theta(\theta) = \frac{1}{2\pi}, \quad 0 \leq \theta < 2\pi, \tag{7.67}$$

## 7.5.2 Rice distribution

Assume $X$ is from $N(\mu_X, \sigma^2)$ and $Y$ is from $N(\mu_Y, \sigma^2)$ . Then the PDF of $R$ of (7.59) is

$$f_R(r) = \frac{r e^{-\frac{r^2+\mu^2}{2\sigma^2}}}{\sigma^2} I_0\left(\frac{r\mu}{\sigma^2}\right), \quad r \geq 0, \tag{7.75}$$

(Note: a typo in (7.75) of the book)

Stephen O. Rice (1907-1986)

which is **Rice distribution** or **Rician distribution**.

where

$$\mu = \sqrt{\mu_X^2 + \mu_Y^2}, \tag{7.76}$$

$$I_0(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos\phi}\, d\phi, \quad -\infty < x < \infty, \tag{7.77}$$

which is the **modified Bessel function of the first kind and zeroth order**.

See Eqs. (7.78), (7.79) and (7.80) of pp. 170-171 to derive (7.75)

### ❖ The normalized Rice distribution

Let the amplitude $R$ be normalized by $\sigma$, i.e.,

$$V = R/\sigma$$

and let

$$m = \mu/\sigma$$

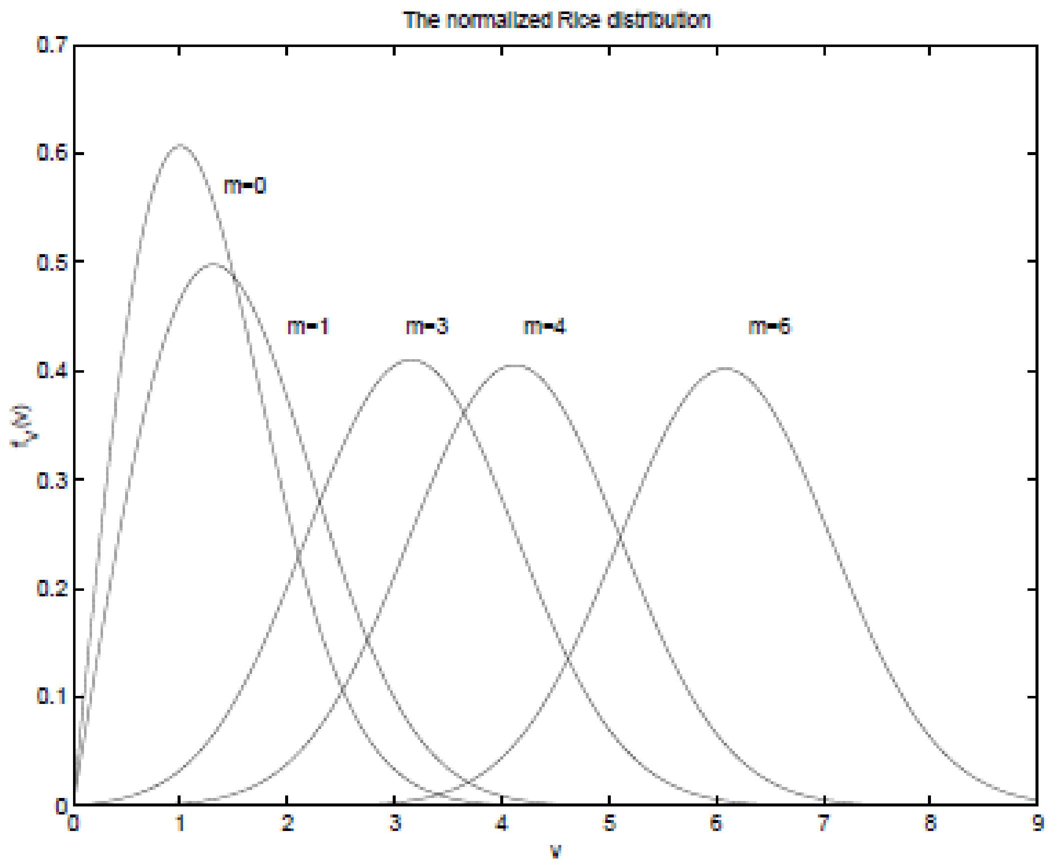



**Figure 7.4** The normalized Rice distribution for $m = 0, 1, 3, 4,$ and $6$.

❖ **The modified Bessel function of the first kind and zeroth order**

Apply the Taylor series expansion to $e^{x\cos\phi}$ In (7.77) and noting $I_0(x)$ is an even function,

$$I_0(x) = \sum_{m=0}^{\infty} \left( \frac{\left(\frac{x}{2}\right)^m}{m!} \right)^2 . \tag{7.83}$$

$$I_0(x) \approx 1 + \frac{x^2}{4}, \ \text{for } x \approx 0, \tag{7.84}$$

$$I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}}, \ \text{for } x \gg 1. \tag{7.85}$$

**Note**: For the modified Bessel function of the first kind and nth order, see http://mathworld.wolfram.com/ModifiedBesselFunctionoftheFirstKind.html

# Part II

## Transform Methods, Bounds and Limits

**Table 8.1.** Four different types of transform methods

| Transform | Definition | Inverse transform |
|---|---|---|
| Moment generating function | $M_X(t) = E\left[e^{tX}\right]$ | $E\left[X^n\right] = M_X^{(n)}(0)$ |
| Characteristic function | $\phi_X(u) = E\left[e^{iuX}\right]$ | $f_X(x) = \mathcal{F}\{\phi_X(u)\}$ |
| Probability generating function | $P_X(z) = E\left[z^X\right]$ | $p_k = \frac{P_X^{(k)}(0)}{k!}$ |
| Laplace transform | $\Phi_X(s) = E\left[e^{-sX}\right]$ | $f_X(x) = \mathcal{L}^{-1}\{\Phi_X(s)\}$ |

# 8 Moment-generating function and characteristic function

## 8.1 Moment-generating function (MGF)

### 8.1.1 Moment-generating function of one random variable

The **moment-generating function** (**MGF**) of a RV X is defined by

$$\boxed{M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx}\, dF_X(x), \ \ t \in I,} \tag{8.1}$$

$$e^{tx} = 1 + tx + \frac{t^2x^2}{2!} + \cdots + \frac{t^nx^n}{n!} + \cdots \,, \tag{8.2}$$

$$M_X(t) = 1 + tE[X] + \frac{t^2}{2!}E[X^2] + \cdots + \frac{t^n}{n!}E[X^n] + \cdots \,. \tag{8.3}$$

$$\left.\frac{d^n}{dt^n}M_X(t)\right|_{t=0} = \frac{n!}{n!}E[X^n], \tag{8.4}$$

$$E[X^n] = M_X^{(n)}(0), \quad n = 0, 1, 2, \ldots, \tag{8.5}$$

The natural logarithm of the MGF

$$m_X(t) = \ln M_X(t), \quad t \in I, \tag{8.6}$$

is called the **logarithmic MGF** (log-MGF) or the **cumulant MGF**.

$$m'_X(0) = \left.\frac{M'_X(t)}{M_X(0)}\right|_{t=0} = E[X] \tag{8.7}$$

$$m''_X(0) = \left.\frac{M''_X(t)M_X(t) - (M'_X(t))^2}{M_X^2(t)}\right|_{t=0} = E[X^2] - (E[X])^2 = \sigma_X^2. \tag{8.8}$$

**Example 8.1: MGF of the binomial distribution**

$$B(k;n,p) = \binom{n}{k} p^k q^{n-k}, \quad k = 0, 1, 2, \ldots, n, \qquad (8.9)$$

$$\boxed{M_X(t) = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k q^{n-k} = \left(p\, e^t + q\right)^n, \quad -\infty < t < \infty,} \qquad (8.10)$$

$$E[X] = M'_X(0) = n(p\,e^t + q)^{n-1} p\,e^t \Big|_{t=0} = np, \qquad (8.12)$$

$$E[X^2] = M''_X(0) = n^2p^2 + npq. \qquad (8.13)$$

$$\text{Var}[X] = E[X^2] - E^2[X] = npq. \qquad (8.14)$$

$$m_X(t) = n \ln\left(p\,e^t + q\right), \quad -\infty < t < \infty. \qquad (8.11)$$

$$E[X] = m'_X(0) = n \frac{p\,e^t}{p\,e^t + q}\Big|_{t=0} = np, \qquad (8.15)$$

$$\sigma_X^2 = m''_X(0) = n \frac{p\,e^t(p\,e^t + q) - (p\,e^t)^2}{(p\,e^t + q)^2}\Big|_{t=0} = npq. \qquad (8.16)$$

## Example 8.2: MGF of Poisson distribution

$$P(k;\lambda) = \frac{\lambda^k}{k!}e^{-\lambda}, \quad k = 0, 1, 2, \ldots \tag{8.17}$$

$$M_X(t) = E[e^{tX}] = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda e^t)^k}{k!} = e^{\lambda(e^t-1)}. \tag{8.18}$$

$$\boxed{m_X(t) = \lambda(e^t - 1), \quad -\infty < t < \infty,} \tag{8.19}$$

$$E[X] = m_X'(0) = \lambda e^t\Big|_{t=0} = \lambda, \tag{8.20}$$

$$\mathrm{Var}[X] = m_X''(0) = \lambda e^t\Big|_{t=0} = \lambda. \tag{8.21}$$

## Example 8.3: MGF of the normal distribution

$$f_U(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}, \quad -\infty < u < \infty. \tag{8.22}$$

$$M_U(t) = E[e^{tU}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tu} e^{-\frac{u^2}{2}}\, du = e^{t^2/2} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(u-t)^2/2}\, du \right). \tag{8.23}$$

$$M_U(t) = e^{\frac{t^2}{2}} \tag{8.24}$$

$$\boxed{m_U(t) = \frac{t^2}{2}, \quad -\infty < t < \infty,} \tag{8.25}$$

$$E[U] = m'_U(0) = 0, \tag{8.26}$$

$$\mathrm{Var}[U] = m''_U(0) = 1. \tag{8.27}$$

$$X = \mu + \sigma U, \tag{8.28}$$

$$M_X(t) = E\left[e^{t(\mu+\sigma U)}\right] = e^{t\mu} E[e^{t\sigma U}] = e^{t\mu} M_U(t\sigma) = \exp\left\{ t\mu + \frac{(t\sigma)^2}{2} \right\}. \tag{8.29}$$

$$m_X(t) = t\mu + \frac{(t\sigma)^2}{2}, \quad -\infty < t < \infty, \tag{8.30}$$

$$E[X] = m'_X(0) = \mu \text{ and } \mathrm{Var}[X] = m''_X(0) = \sigma^2.$$

The *n*th **central moment**

$$E[(X-\mu)^n] = \sigma^n E[U^n], \quad n = 1, 2, \ldots. \tag{8.31}$$

By applying the Taylor-Series expansion to $M_U(t) = e^{t^2/2}$

$$M_U(t) = 1 + \frac{t^2}{2} + \frac{t^4}{8} + \cdots + \frac{1}{2^k k!} t^{2k} + \cdots. \tag{8.32}$$

$$M_U(t) = 1 + tE[U] + \frac{E[U^2]}{2} t^2 + \cdots + \frac{E[U^n]}{n!} t^n + \cdots. \tag{8.33}$$

$$E[U^n] = \begin{cases} 0, & n \text{ odd}, \\ 1 \cdot 3 \cdot 5 \cdots (n-3)(n-1), & n \text{ even}. \end{cases} \tag{8.35}$$

## 8.1.2 Moment-generating function of sum of independent random variables

Let $Y = X_1 + X_2$, where $X_1$ and $X_2$ are independent. Then the MGF of $Y$ is

$$M_Y(t) = E[e^{tY}] = E[e^{tX_1}]E[e^{tX_2}] = M_{X_1}(t)M_{X_2}(t), \tag{8.37}$$

Let $X_i$, $i = 1, 2, \ldots, m$, be $m$ independent RVs with corresponding MGFs $M_{X_i}(t)$.

Define $Y$ as their sum.

$$Y = \sum_{i=1}^{m} X_i. \tag{8.38}$$

Then the MGF of $Y$ is

$$\boxed{M_Y(t) = \prod_{i=1}^{m} M_{X_i}(t), \ \ t \in I,} \tag{8.39}$$

### 8.1.3 Joint moment-generating function of multivariate random variables

$\boldsymbol{X} = (X_1, X_2, \ldots, X_m)^\top$ is an $m$-dimensional vector random variable

The joint MGF

$$M_{\boldsymbol{X}}(\boldsymbol{t}) \triangleq E\left[e^{t_1 X_1 + t_2 X_2 + \cdots + t_m X_m}\right] = E\left[e^{\langle \boldsymbol{t}, \boldsymbol{x} \rangle}\right], \text{ for } \boldsymbol{t} \in \boldsymbol{I}, \quad (8.40)$$

$$E\left[X_1^{n_1} \cdots X_m^{n_m}\right] = \left. \frac{\partial^{n_1 + \cdots + n_m} M_{\boldsymbol{X}}(\boldsymbol{t})}{\partial t_1^{n_1} \cdots \partial t_m^{n_m}} \right|_{\boldsymbol{t}=\boldsymbol{0}} \quad (8.41)$$

**Example 8.4: Joint MGFs of bivariate and multivariate normal distributions**

bivariate normal variables $\boldsymbol{X} = (X_1, X_2)^\top$

$$Y = t_1 X_1 + t_2 X_2 = \boldsymbol{t}^\top \boldsymbol{X} \triangleq \langle \boldsymbol{t}, \boldsymbol{X} \rangle. \quad (8.42)$$

$$\mu_Y = t_1 \mu_1 + t_2 \mu_1 = \langle \boldsymbol{t}, \boldsymbol{\mu} \rangle \quad (8.43)$$

$$\begin{aligned}\sigma_Y^2 &= E[(Y - \mu_Y)^2] \\ &= t_1^2 E[(X_1 - \mu_1)^2] + 2 t_1 t_2 E[(X_1 - \mu_1)(X_2 - \mu_2)] + t_2^2 E[(X_2 - \mu_2)^2] \\ &= t_1^2 \sigma_1^2 + 2\rho t_1 t_2 \sigma_1 \sigma_2 + t_2^2 \sigma_2^2 = \langle \boldsymbol{t}, \boldsymbol{t} \boldsymbol{C} \rangle = \boldsymbol{t}^\top \boldsymbol{C} \boldsymbol{t}, \quad (8.44)\end{aligned}$$

where

$$C = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}. \tag{8.46}$$

Writing

$$Y = \sigma_Y U + \mu_Y,$$

$$\begin{aligned} M_Y(\xi) &= E[e^{\xi Y}] = e^{\xi\mu_Y} E[e^{\sigma_Y \xi U}] \\ &= e^{\xi\mu_Y} M_U(\sigma_Y \xi) = e^{\xi\mu_Y} e^{\frac{(\sigma_Y \xi)^2}{2}}, \end{aligned} \tag{8.47}$$

If we set $\xi = 1$ :

$$M_Y(1) = E[e^Y] = e^{\mu_Y + \frac{\sigma_Y^2}{2}}. \tag{8.48}$$

From the definition of the joint MGF

$$E[e^Y] = E[e^{\langle \boldsymbol{t}, \boldsymbol{X} \rangle}] = M_{\boldsymbol{X}}(\boldsymbol{t}). \tag{8.49}$$

Thus, we find the joint MGF:

$$M_{\boldsymbol{X}}(\boldsymbol{t}) = e^{\mu_Y + \frac{\sigma_Y^2}{2}} = \exp\left\{ \boldsymbol{t}^\top \boldsymbol{\mu} + \frac{\boldsymbol{t}^\top C \boldsymbol{t}}{2} \right\}. \tag{8.50}$$

## ❖ Generalization to a multivariate normal distribution

$$\boldsymbol{X} = (X_1, X_2, \ldots, X_m)^\top$$

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{(2\pi)^{m/2}|\det \boldsymbol{C}|^{1/2}} \exp\left\{-\frac{(\boldsymbol{x}-\boldsymbol{\mu})^\top \boldsymbol{C}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}{2}\right\}, \tag{8.52}$$

$$\boldsymbol{C} = E[(\boldsymbol{X}-\boldsymbol{\mu})(\boldsymbol{X}-\boldsymbol{\mu})^\top] = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1m}\sigma_1\sigma_m \\ \rho_{21}\sigma_2\sigma_1 & \sigma_2^2 & \cdots & \rho_{2m}\sigma_2\sigma_m \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{m1}\sigma_m\sigma_1 & \rho_{m2}\sigma_m\sigma_2 & \cdots & \sigma_m^2 \end{bmatrix}. \tag{8.53}$$

$$M_{\boldsymbol{X}}(\boldsymbol{t}) = \exp\left\{\boldsymbol{t}^\top\boldsymbol{\mu} + \frac{\boldsymbol{t}^\top\boldsymbol{C}\boldsymbol{t}}{2}\right\}, \quad \boldsymbol{t} \in \mathbb{R}^m. \tag{8.54}$$

## 8.2 Characteristic Function (CF)

$$\phi_X(u) = E[e^{iuX}] = \int_{-\infty}^{\infty} e^{iux}\, dF_X(x), \quad -\infty < u < \infty, \tag{8.55}$$

For a continuous RV

$$\phi_X(u) = E[e^{iuX}] = \int_{-\infty}^{\infty} e^{iuX} f_X(x)\, dx, \tag{8.56}$$

For a discrete RV

$$\phi_X(u) = E[e^{iuX}] = \sum_k e^{iux_k} P[x_k]. \tag{8.57}$$

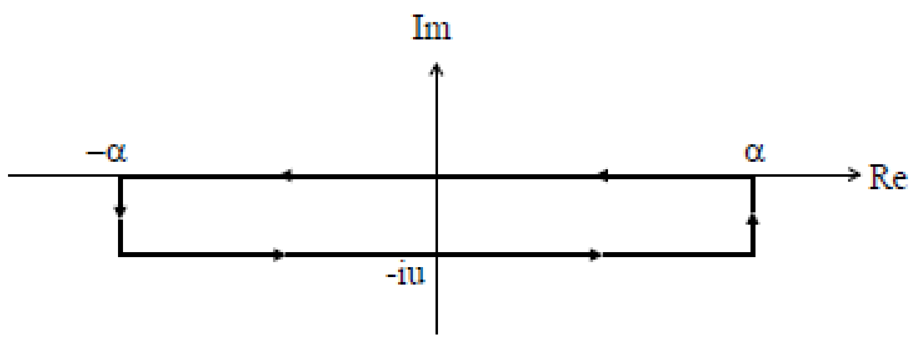
**Example 8.5:** **Characteristic function of the normal distribution**

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \qquad -\infty < x < \infty.$$

$$\phi_X(u) = E[e^{iuX}] = \int_{-\infty}^{\infty} e^{iux} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx = \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x-iu)^2}{2}\right\} dx. \qquad (8.63)$$

We make the change of variable $s = x - iu$

$$\phi_X(u) = \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \int_{-\infty-iu}^{\infty-iu} e^{-\frac{s^2}{2}} ds.$$




(a) for $u > 0$

The function $e^{-\frac{s^2}{2}}$ is **analytic** (i.e., possessing no poles), the integral around the contour of Figure (a) is zero – the **Cauchy-Goursat integral theorem**.

$$\int_{-\alpha-iu}^{\alpha-iu} e^{-\frac{s^2}{2}}\,ds + \int_{\alpha-iu}^{\alpha} e^{-\frac{s^2}{2}}\,ds + \int_{\alpha}^{-\alpha} e^{-\frac{x^2}{2}}\,dx + \int_{-\alpha}^{-\alpha-iu} e^{-\frac{s^2}{2}}\,ds = 0 \qquad (8.65)$$

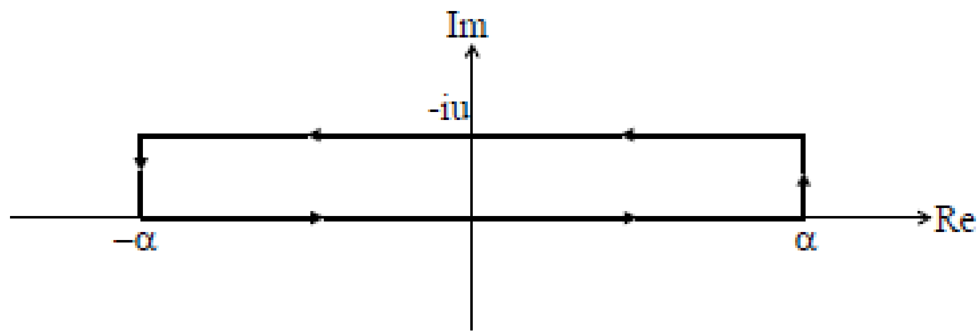
The second and fourth terms approach zero as $\alpha \to \infty$,

$$\int_{-\infty-iu}^{\infty-iu} e^{-\frac{s^2}{2}}\,ds - \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx = 0. \qquad (8.66)$$

The second term is $\sqrt{2\pi}$, so must be the second term. Hence from (8.64)

$$\boxed{\phi_X(u) = e^{-\frac{u^2}{2}}, \quad u > 0.} \qquad (8.67)$$

For the case u < 0, the contour integral in Figure (b) will lead to the same result (Problem 8.14).




(b) for $u < 0$

- By applying the transformation *Y*=(*X*-μ)/σ , we find the CF of *N*(μ, σ²)

$$\phi_X(u) = E[e^{iu(\mu+\sigma Y)}] = e^{iu\mu}E[e^{i(u\sigma)Y}]$$
$$= e^{iu\mu}\phi_Y(u\sigma) = \exp\left\{iu\mu - \frac{(u\sigma)^2}{2}\right\}, \quad -\infty < u < \infty, \qquad (8.68)$$

- The cumulative generating function (CGF)

$$\psi_X(u) \triangleq \ln \phi_X(u) = iu\mu - \frac{(u\sigma)^2}{2}. \qquad (8.69)$$

$$\mu_X = (-i)\psi'_X(0) = \mu, \text{ and } \sigma_X^2 = (-i)^2\psi''_X(0) = \sigma^2. \qquad (8.70)$$

### 8.2.2 Sum of independent random variables and convolution

$Y = X_1 + X_2$

$$\phi_Y(u) = E[e^{iuY}] = E[e^{iuX_1}]E[e^{iuX_2}] = \phi_{X_1}(u)\phi_{X_2}(u), \quad -\infty < u < \infty, \tag{8.71}$$

$$f_Y(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_{X_1}(u)\phi_{X_2}(u)e^{-iuy}\,du. \tag{8.72}$$

By substituting $\phi_{X_1}(u)$ of (8.56) into the above,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X_1}(x)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_{X_2}(u)e^{-iu(y-x)}\,du\right\}dx, \tag{8.73}$$

which, with (8.59), leads to

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X_1}(x)f_{X_2}(y-x)\,dx. \tag{8.74}$$

$$\boxed{f_Y(y) = f_{X_1}(y) \circledast f_{X_2}(y).} \tag{8.75}$$

The above is called the **convolution integral** (or simply **convolution**) of $f_{X_1}(\cdot)$ and $f_{X_2}(\cdot)$

**Example 8.6: Sum of independent normal variables**

Let $Y = X_1 + X_2$.

$$\phi_Y(u) = \phi_{X_1}(u)\phi_{X_2}(u) = \exp\left\{iu(\mu_1 + \mu_2) - \frac{u^2(\sigma_1^2 + \sigma_2^2)}{2}\right\}. \quad (8.79)$$

Thus, we see the RV $Y$ is also a normal variable with distribution

$$\mu = \mu_1 + \mu_2 \text{ and } \sigma^2 = \sigma_1^2 + \sigma_2^2.$$

This reproductive property of normal variables holds for the sum of any number of independent normal variables

### 8.2.3 Moment generation from characteristic function

$$\phi_X^{(n)}(u) \triangleq \frac{d^n \phi_X(u)}{du^n} = \int_{-\infty}^{\infty} (ix)^n e^{iux} f_X(x)\, dx, \quad -\infty < u < \infty. \quad (8.80)$$

$$\boxed{E[X^n] = (-i)^n \phi_X^{(n)}(0), \text{ for } n = 1, 2, \ldots.} \quad (8.81)$$

Assuming that the Taylor-series expansion of the CF exists throughout some interval in *u* that includes the origin,

$$\phi_X(u) = \sum_{n=0}^{\infty} \frac{\phi_X^{(n)}(0)u^n}{n!}. \qquad (8.82)$$

Using (8.81),

$$\boxed{\phi_X(u) = \sum_{n=0}^{\infty} E[X^n]\frac{(iu)^n}{n!}, \quad -\infty < u < \infty.} \qquad (8.83)$$

The cumulant generating function (CGN) defined in (8.69)

$$\psi_X(u) = \ln \phi_X(u).$$

may also be expanded:

$$\psi_X(u) = \sum_{n=0}^{\infty} \kappa_n \frac{(iu)^n}{n!}. \qquad (8.85)$$

The quantities $\kappa_n$ are called **cumulants**.

$$\kappa_1 = (-i)\psi'_X(0) = \mu, \text{ and } \kappa_2 = (-i)^2\psi''_X(0) = \sigma^2. \qquad (8.86)$$

## 8.2.4 Joint characteristic function of multivariate random variables

$\boldsymbol{X} = (X_1, X_2, \ldots, X_m)^\top$ is an $m$-dimensional random vector

We define the **joint CF** as

$$\phi_{\boldsymbol{X}}(\boldsymbol{u}) = E\left[e^{i(u_1X_1+u_2X_2+\cdots+u_mX_m)}\right]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp\{i\langle \boldsymbol{u}, \boldsymbol{x}\rangle\} f_{\boldsymbol{x}}(\boldsymbol{x})\, dx_1\, dx_2 \cdots dx_m, \qquad (8.87)$$

The joint moment, if it exists, can be obtained by

$$E\left[X_1^{n_1}\cdots X_m^{n_m}\right] = (-i)^{n_1+\cdots+n_m}\left[\frac{\partial^{n_1+\cdots+n_m}\phi_{\boldsymbol{X}}(\boldsymbol{u})}{\partial u_1^{n_1}\cdots\partial u_m^{n_m}}\right]_{\boldsymbol{u}=\boldsymbol{0}} \qquad (8.88)$$

The inverse transform (8.59) can be extended to the multivariate case.

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \left(\frac{1}{2\pi}\right)^m \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp\{-i\langle \boldsymbol{u}, \boldsymbol{x}\rangle\}\phi_{\boldsymbol{x}}(\boldsymbol{x})\, du_1\, du_2\cdots du_m. \qquad (8.89)$$

**Example 8.7:** **Bivariate normal distribution.**

$$Y = \langle \boldsymbol{u}, \boldsymbol{X} \rangle \tag{8.90}$$

$$\mu_Y = \langle \boldsymbol{u}, \boldsymbol{\mu} \rangle \tag{8.91}$$

$$\sigma_Y^2 = \boldsymbol{u}^\top C \boldsymbol{u} \tag{8.92}$$

By writing $Y = \sigma_Y V + \mu_Y$

$$\phi_Y(u) = E\left[e^{iuY}\right] = e^{iu\mu_Y} E\left[e^{iu\sigma_Y V}\right] = e^{iu\mu_Y} e^{-\frac{(u\sigma_Y)^2}{2}}, \tag{8.93}$$

If we set $u = 1$

$$\phi_Y(1) = E\left[e^{iY}\right] = e^{i\mu_Y - \frac{\sigma_Y^2}{2}} \tag{8.94}$$

But

$$E\left[e^{iY}\right] = E\left[e^{i\langle \boldsymbol{u}, \boldsymbol{X} \rangle}\right] = \phi_{\boldsymbol{X}}(\boldsymbol{u}). \tag{8.95}$$

Thus,

$$\boxed{\phi_{\mathbf{X}}(\mathbf{u}) = \exp\left\{ i\langle \boldsymbol{u}, \boldsymbol{\mu} \rangle - \frac{\boldsymbol{u}^\top C \boldsymbol{u}}{2} \right\}.} \tag{8.96}$$

The formula (8.96) holds for the multivariate normal variables as well (Table 8.2).

## 8.2.5 Application of the Characteristic Function: The Central Limit Theorem (CLT)

Let $\{X_i ; 1 \le i \le n\}$ be n independent samples from a population with an arbitrary Distribution function $F(x)$, but with finite mean μ and variance $\sigma^2$.

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad (8.97)$$

**Theorem 8.2 (The central limit theorem).** *If $\overline{X}_n$ is the average of $n$ **independent samples** from a distribution having finite variance $\sigma^2$ and mean $\mu$, then*

$$\boxed{\lim_{n\to\infty} P\left[\frac{\sqrt{n}}{\sigma}(\overline{X}_n - \mu) \le x\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{u^2}{2}}\, du\,.} \qquad (8.98)$$

*Thus, $\overline{X}_n$ is **asymptotically normally distributed** according to $N(\mu, \sigma^2/n)$.*

*Proof.* Let $\phi_n(u)$ be the CF of the RV $\sqrt{n}(\overline{X}_n - \mu)/\sigma$:

$$\phi_n(u) = E[e^{iu\sqrt{n}(\overline{X}_n - \mu)/\sigma}] = E\left[\exp\left\{iu\sum_{i=1}^{n}\frac{(X_i - \mu)}{\sqrt{n}\sigma}\right\}\right] = (\phi(u))^n\,, \qquad (8.99)$$

where $\phi(u)$ is the CF of $(X_i - \mu)/\sqrt{n}\sigma$, common to all $X_i$'s

By applying the Taylor series expansion (see (8.83))

$$\phi(u) = 1 + iE\left[\frac{X-\mu}{\sqrt{n}\sigma}\right]u - \frac{1}{2}E\left[\left(\frac{X-\mu}{\sqrt{n}\sigma}\right)^2\right]u^2 + o\left(\frac{u^2}{n}\right), \qquad (8.100)$$

Thus,

$$\lim_{n\to\infty}\phi_n(u) = \lim_{n\to\infty}\left[1 - \frac{u^2}{2n} + o\left(\frac{u^2}{n}\right)\right]^n = e^{-\frac{u^2}{2}}, \qquad (8.101)$$

Thus, the distribution function of the RV $\sqrt{n}\sigma^{-1}(\overline{X}_n - \mu)$ converges to that of the distribution N(0, 1):

$$\lim_{n\to\infty} P\left[\frac{\sqrt{n}}{\sigma}(\overline{X}_n - \mu) \le x\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{u^2}{2}}\,du. \qquad (8.102)$$

Thus, $\overline{X}_n$ is **asymptotically normally distributed** according to $N(\mu, \sigma^2/n)$.