Lecture 15
Discrete-Time Markov Chains (DTMC)

ELE 525: Random Processes in Information Systems

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Textbook: Hisashi Kobayashi, Brian L. Mark and William Turin, ***Probability, Random Processes and Statistical Analysis*** (Cambridge University Press, 2012)

15 Discrete-Time Markov Chains

15.1 Markov Processes and Markov Chains

Definition 15.1 (Markov process). *A random process $X(t)$ is called a **Markov process** if for any $t_1 < t_2$*

$$P[X(t_2) \leq x | X(t); -\infty < t \leq t_1] = P[X(t_2) \leq x | X(t_1)]. \tag{15.1}$$

□

Consider a set of n arbitrarily chosen instants $t_1 < t_2 < \ldots < t_{n-1} < t_n$

$$P[X(t_n) \leq x | X(t_i); i = 1, 2, \ldots, n-1] = P[X(t_n) \leq x | X(t_{n-1})]. \tag{15.2}$$

Definition 15.2 (h-th order Markov chain). *A random sequence[1] X_n is called a **Markov chain** of **order h** (or an h-th order Markov chain), when X_n takes a finite or countably infinite number of states $\mathcal{S} = \{0, 1, 2, \ldots\}$, and if the state of X_n depends only on the last h states, i.e., if*

$$P[X_n | X_k; -\infty < k \leq n-1] = P[X_n | X_{n-h}, \ldots, X_{n-1}], \text{ for all } n. \tag{15.4}$$

*A first-order Markov chain is called a **simple** Markov chain.* □

Definition 15.3 (Transition probabilities). *The conditional probabilities of a (simple) Markov chain*

$$P[X_{n+1} = j | X_n = i] = P_{ij}(n), \quad i, j \in \mathcal{S}, \quad n = 0, 1, 2, \ldots \tag{15.7}$$

are called the ***(one-step) transition probabilities****. The matrix*

$$\boldsymbol{P}(n) = [P_{ij}(n)] \tag{15.8}$$

is called the ***transition probability matrix*** *(TPM).* □

Example 15.1: Random walk.

$$P[X_n = +1] = p, \quad P[X_n = -1] = 1 - p \triangleq q.$$

$$S_n = X_1 + X_2 + \ldots + X_n = S_{n-1} + X_n$$

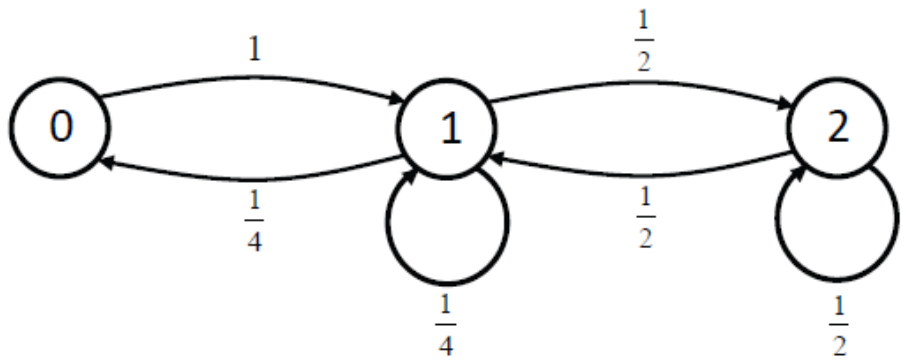
$$P_{ij}(n) = \begin{cases} p, & j = i + 1, \\ q, & j = i - 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$p_j(n+1) = \sum_{i \in \mathcal{S}} p_i(n) P_{ij}(n), \quad j \in \mathcal{S}, \tag{15.10}$$

If all $P_{ij}(n)$ are independent of n,

$$\boxed{\boldsymbol{p}^\top(n+1) = \boldsymbol{p}^\top(n)\boldsymbol{P} \ \text{ for all } n = 0, 1, 2, \ldots,} \tag{15.11}$$

called a **homogeneous** or **stationary** Markov chain.



$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$P_{ij}(n) \geq 0 \quad \text{for all } i,\ j, \quad \text{and} \quad n = 0, 1, 2, \ldots \tag{15.13}$$

$$\sum_{j \in \mathcal{S}} P_{ij}(n) = 1 \quad \text{for all } i \in \mathcal{S}, \quad \text{and} \quad n = 0, 1, 2, \ldots. \tag{15.14}$$

Matrices satisfying (15.13) and (15.14) are called **stochastic**.

Any stochastic square matrix may serve as a TPM.

By applying (15.11) recursively, we have

$$\boldsymbol{p}^\top(n) = \boldsymbol{p}^\top(0)\boldsymbol{P}(0)\boldsymbol{P}(1)\cdots\boldsymbol{P}(n-1), \tag{15.15}$$

If the Markov chain is time-homogeneous or stationary,

$$\boldsymbol{p}^\top(n) = \boldsymbol{p}^\top(0)\boldsymbol{P}^n, \tag{15.16}$$

If we set $n \leftarrow m + n$

$$\boldsymbol{p}^\top(m+n) = \boldsymbol{p}^\top(0)\boldsymbol{P}^m\boldsymbol{P}^n = \boldsymbol{p}^\top(m)\boldsymbol{P}^n. \tag{15.17}$$

By denoting

$$\boldsymbol{P}^n = \left[P_{ij}^{(n)}\right], \quad i, j \in \mathcal{S}, \tag{15.18}$$

$$p_j(m+n) = \sum_{i \in \mathcal{S}} p_i(m) P_{ij}^{(n)}, \text{ where } i, j \in \mathcal{S}, \; m, n = 0, 1, 2, \ldots. \tag{15.19}$$

Theorem 15.1 (Chapman-Kolmogorov Equations). *In a homogeneous Markov chain the following equations hold for all states $i, j, k \in \mathcal{S}$ and time (or step) indices $m, n = 0, 1, 2 \ldots$:*

$$\boxed{P_{ik}^{(m+n)} = \sum_{j \in \mathcal{S}} P_{ij}^{(m)} P_{jk}^{(n)}.} \tag{15.20}$$

Proof. The above result is immediately obtainable from

$$\boldsymbol{P}^{m+n} = \boldsymbol{P}^m \boldsymbol{P}^n. \tag{15.21}$$

Since the Markov chain is homogeneous,

$$\begin{aligned}
P_{ik}^{(m+n)} &= P[X_{m+n} = k | X_0 = i] \\
&= \sum_{j \in \mathcal{S}} P[X_{m+n} = k, X_m = j | X_0 = i] \\
&\overset{(a)}{=} \sum_{j \in \mathcal{S}} P[X_{m+n} = k | X_m = j, X_0 = i] P[X_m = j | X_0 = i] \\
&\overset{(b)}{=} \sum_{j \in \mathcal{S}} P[X_{m+n} = k | X_m = j] P[X_m = j | X_0 = i] \\
&= \sum_{j \in \mathcal{S}} P_{jk}^{(n)} P_{ij}^{(m)},
\end{aligned} \tag{15.22}$$

(a) is obtained using the formula $P[A \cap B | C] = P[A | B \cap C] P[B | C]$
(b) makes use of the Markov property

15.2 Computation of State Probabilities

15.2.1 Generating Function Method

$$\boxed{\boldsymbol{g}(z) = \sum_{n=0}^{\infty} \boldsymbol{p}(n) z^n,} \tag{15.23}$$

$$\begin{aligned} \boldsymbol{g}^\top(z)\boldsymbol{P} &= \sum_{n=0}^{\infty} \boldsymbol{p}^\top(n+1) z^n = z^{-1} \sum_{n=0}^{\infty} \boldsymbol{p}^\top(n+1) z^{n+1} \\ &= z^{-1}\boldsymbol{g}^\top(z) \; - \; z^{-1}\boldsymbol{p}^\top(0), \end{aligned} \tag{15.24}$$

$$\boxed{\boldsymbol{g}^\top(z) = \boldsymbol{p}^\top(0)[\boldsymbol{I} - \boldsymbol{P}z]^{-1},} \tag{15.25}$$

15.2.2 Spectral Expansion Method

$$\boldsymbol{P}\boldsymbol{u}_i = \lambda_i \boldsymbol{u}_i, \quad i \in \mathcal{S} = \{0, 1, 2, \ldots, M-1\}, \tag{15.33}$$

$$\boldsymbol{U} = [\boldsymbol{u}_0 \boldsymbol{u}_1 \cdots \boldsymbol{u}_{M-1}] \tag{15.34}$$

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{M-1} \end{bmatrix}. \tag{15.35}$$

$$\boxed{\boldsymbol{P} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{-1} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{V},} \tag{15.36}$$

$$\boldsymbol{V} = \boldsymbol{U}^{-1}. \tag{15.37}$$

By multiplying $\boldsymbol{V}$ on the left of (15.36),

$$\boldsymbol{V}\boldsymbol{P} = \boldsymbol{\Lambda}\boldsymbol{V}. \tag{15.38}$$

Defining a set of row vectors $\boldsymbol{v}_i^\top$s by

$$\boldsymbol{V} = \begin{bmatrix} \boldsymbol{v}_0^\top \\ \boldsymbol{v}_1^\top \\ \vdots \\ \boldsymbol{v}_{M-1}^\top \end{bmatrix}, \tag{15.39}$$

we find from (15.38)

$$\boldsymbol{v}_i^\top \boldsymbol{P} = \lambda_i \boldsymbol{v}_i^\top, \quad i \in \mathcal{S}. \tag{15.40}$$

From (15. 37)

$$\boldsymbol{V}\boldsymbol{U} = \boldsymbol{I}, \tag{15.41}$$

$\boldsymbol{v}_i$ and $\boldsymbol{u}_j$ are **bi-orthonormal**

$$\boxed{\boldsymbol{v}_i^\top \boldsymbol{u}_j = \delta_{ij}, \quad i, j \in \mathcal{S}.} \tag{15.42}$$

From (15.36)

$$\boldsymbol{P}^2 = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{-1}\boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{-1} = \boldsymbol{U}\boldsymbol{\Lambda}^2\boldsymbol{U}^{-1}. \tag{15.43}$$

$$\boxed{\boldsymbol{P}^n = \boldsymbol{U}\boldsymbol{\Lambda}^n\boldsymbol{U}^{-1} = \sum_{i\in\mathcal{S}} \lambda_i^n \boldsymbol{u}_i\boldsymbol{v}_i^\top = \sum_{i\in\mathcal{S}} \lambda_i^n \boldsymbol{E}_i,} \tag{15.44}$$

$$\boldsymbol{E}_i = \boldsymbol{u}_i\boldsymbol{v}_i^\top, \quad i \in \mathcal{S}, \tag{15.45}$$

are the **projection matrices**

$$\sum_{i\in\mathcal{S}} \boldsymbol{E}_i = \sum_{i\in\mathcal{S}} \boldsymbol{E}_i^\top = \boldsymbol{I}, \tag{15.46}$$

$$P_{ij}^{(n)} = \sum_{k\in\mathcal{S}} \lambda_k^n u_{ki} v_{kj}, \quad i,j \in \mathcal{S}, \quad n = 0,1,2,\ldots. \tag{15.47}$$

$$\boldsymbol{p}^\top(n) = \boldsymbol{p}^\top(0)\boldsymbol{P}^n = \sum_{k\in\mathcal{S}} \lambda_k^n \boldsymbol{p}^\top(0)\boldsymbol{u}_k\boldsymbol{v}_k^\top. \tag{15.48}$$

$$p_i(n) = \sum_{k\in\mathcal{S}} \lambda_k^n \left(\boldsymbol{p}^\top(0)\boldsymbol{u}_k\right) v_{ki}, \quad i \in \mathcal{S}. \tag{15.49}$$

15.3 Classification of States

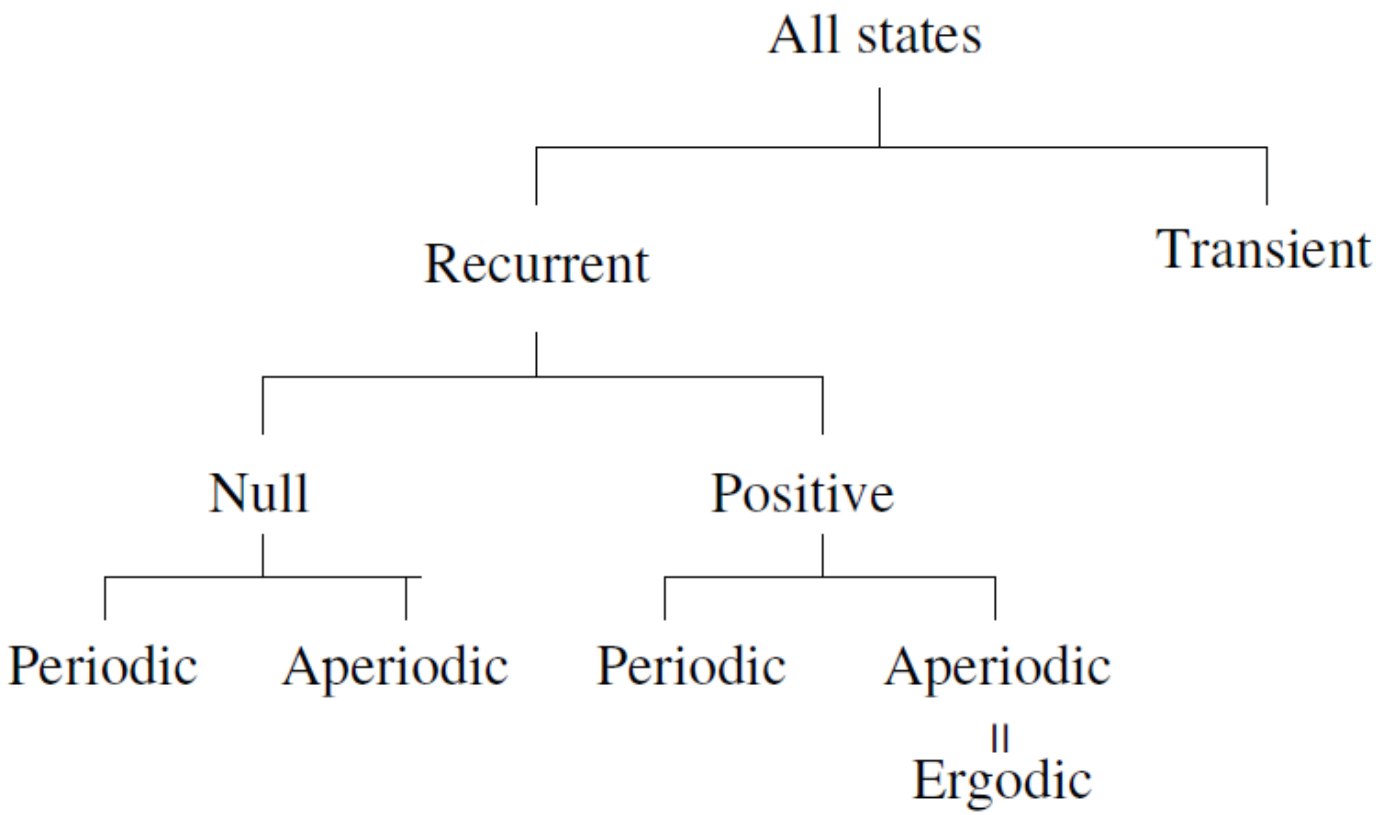


Figure 15.2 Classification of states in a Markov chain.

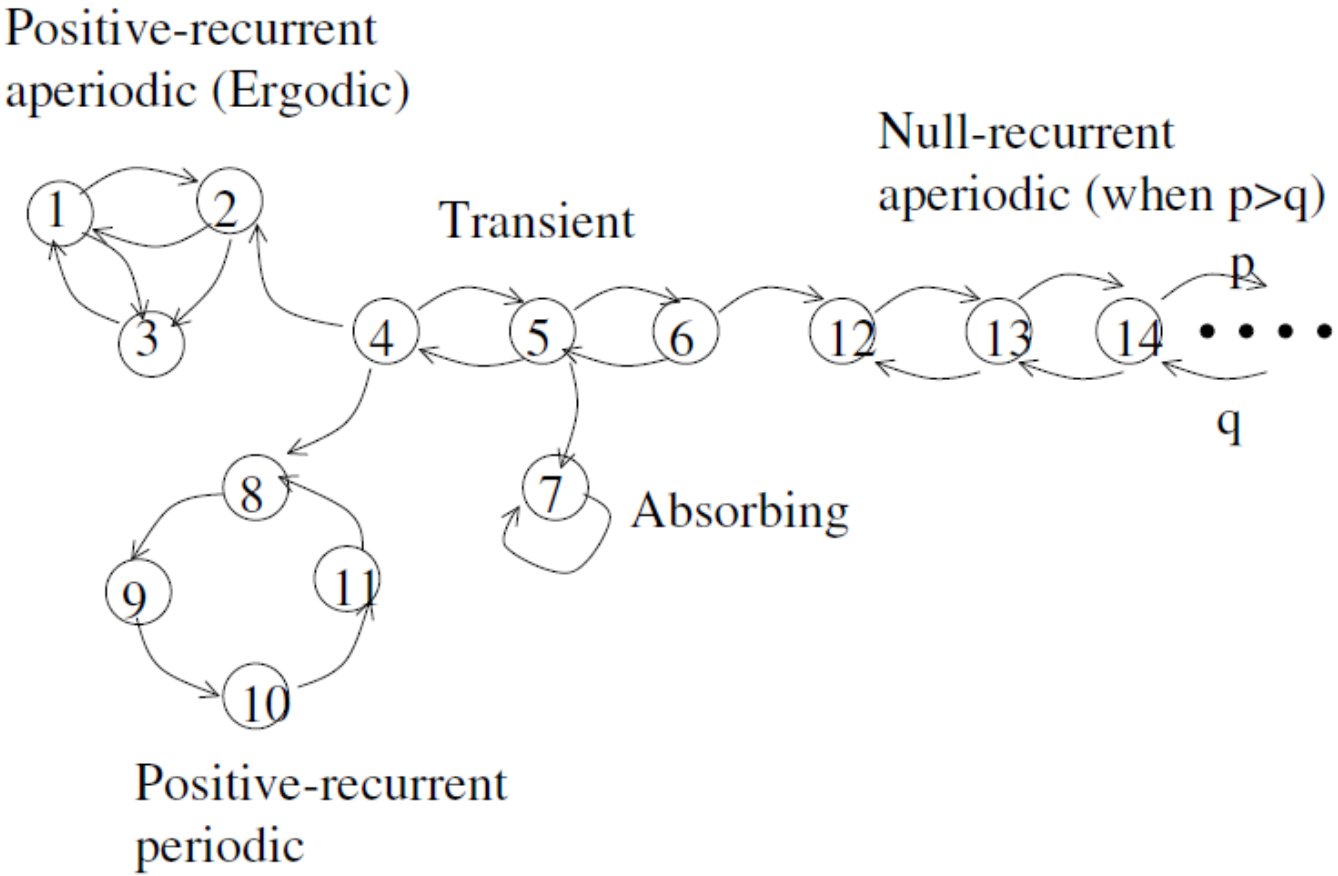


Figure 15.3 An example of a Markov chain with various states.

15.3.1 Recurrent and Transient States

T_{ij} : the **first-passage time** from i to j.

Let $f_{ij}^{(n)}$ be

$$\boxed{f_{ij}^{(n)} \triangleq P[T_{ij} = n], \quad i, j \in \mathcal{S}, \quad n = 1, 2, \ldots.} \tag{15.58}$$

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}, \quad i, j \in \mathcal{S} \tag{15.59}$$

If $f_{ij} = 1$, $\{f_{ij}^{(n)} : n = 1, 2, \ldots\}$ represents the probability distribution of the first-passage time.

if $f_{ij} < 1$, the process may never reach state j.

$$f_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)} \tag{15.60}$$

is the probability that the system ever returns to state i.

Definition 15.4 (Recurrent state and transient state). *State $i \in \mathcal{S}$ is called a* ***recurrent state*** *(or a* ***persistent*** *state) if $f_{ii} = 1$. It is called a* ***transient state*** *if $f_{ii} < 1$.* □

The **mean first-passage time** (or **expected first-passage time**)

$$\boxed{\mu_{ij} = E[T_{ij}] = \begin{cases} \sum_{n=1}^{\infty} n f_{ij}^{(n)}, & \text{if } f_{ij} = 1 \\ \infty, & \text{if } f_{ij} < 1. \end{cases}, \quad i, j \in \mathcal{S}.} \tag{15.61}$$

When $f_{ij} = 1$

$$\boxed{\mu_{ij} = P_{ij} + \sum_{k \neq j} P_{ik}(\mu_{kj} + 1),} \tag{15.62}$$

When j=i, the mean first-passage time is called the **mean recurrence time** (or **expected recurrence time**).

Definition 15.5 (Null-recurrent state and positive-recurrent state). *A recurrent state $i \in \mathcal{S}$ is called a **null-recurrent** state if $\mu_{ii} = \infty$, and is called a **positive-recurrent** (or **regular-recurrent**) state if $\mu_{ii} < \infty$.* □

If $P_{ii}^{(n)} = 0$ for $n \neq d_i, 2d_i, \ldots$, then state i is called **periodic** with period d_i.

$$d_i = \gcd\left\{n : P_{ii}^{(n)} > 0\right\}, \tag{15.63}$$

Definition 15.6 (Periodic state and aperiodic state). *A state $i \in \mathcal{S}$ is called **periodic** with period d_i if $d_i > 1$, and is called **aperiodic** if $d_i = 1$.* □

Definition 15.7 (Ergodic state). *State $i \in \mathcal{S}$ is called an **ergodic** state, if it is positive-recurrent and aperiodic, i.e., $f_{ii} = 1$, $\mu_{ii} < \infty$, and $d_i = 1$.* □

Definition 15.8 (Absorbing state). *State $i \in \mathcal{S}$ is called an **absorbing state** if the transition probability satisfies $P_{ii} = 1$.* □

15.3.2 Criteria for State Classification

$$G_{ij}(z) \triangleq \sum_{n=0}^{\infty} P_{ij}^{(n)} z^n, \quad i, j \in \mathcal{S}, \tag{15.64}$$

$$F_{ij}(z) \triangleq \sum_{n=0}^{\infty} f_{ij}^{(n)} z^n, \quad i, j \in \mathcal{S}. \tag{15.65}$$

Theorem 15.2 (Generating functions of n-step transition and first-passage probabilities). *The generating functions $G_{ij}(z)$ and $F_{ij}(z)$ defined above are related according to*

$$\boxed{G_{ij}(z) = \delta_{ij} + F_{ij}(z) G_{jj}(z), \quad i, j \in \mathcal{S}.} \tag{15.66}$$

Proof. Read p. 441

□

For $i = j$, we obtain

$$\boxed{G_{ii}(z) = \frac{1}{1 - F_{ii}(z)}.} \tag{15.70}$$

The first-passage probability

$$\boxed{f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)} = F_{ij}(1).} \tag{15.71}$$

The mean first-passage time

$$\boxed{\mu_{ij} = \sum_{n=1}^{\infty} n f_{ij}^{(n)} = F'_{ij}(1).} \tag{15.72}$$

Note: We shall skip in the class discussion of Theorem 15.3 (**Conditions for recurrent and transient states**) and Theorem 15.5 (**Conditions for null-recurrent, ergodic or periodic states**) of pp. 442-445, but be aware that these theorems exist.

15.3.3 Communicating States and an Irreducible Markov Chain

Definition 15.9 (Reachable states and communicating states). *We say that state $j \in \mathcal{S}$ is* ***reachable*** *(or* ***accessible****) from state $i \in \mathcal{S}$ if there is an integer $n \geq 1$ such that $P_{ij}^{(n)} > 0$. If state i is reachable from state j and state j is reachable from state i, then the states i and j are said to* ***communicate***[4]*, written as $i \leftrightarrow j$.* □

Theorem 15.6 (Reachable states from a recurrent state). *Suppose state j is reachable from a recurrent state i (i.e., $j \leftarrow i$). Then state i is also reachable from state j ($i \leftarrow j$); hence states i and j communicate ($i \leftrightarrow j$). Moreover, state j is also recurrent.*

Proof. See pp. 445-446.

If two states i and j do not communicate, then either

$$P_{ij}^{(n)} = 0 \text{ for all } n \geq 1,$$

or

$$P_{ji}^{(n)} = 0 \text{ for all } n \geq 1,$$

or both relations are true. The relation of communications "$\leftrightarrow$" is an **equivalence relation**; that is, the following three properties hold:

1. **Reflexive Property**: $i \leftrightarrow i$ for all $i \in \mathcal{S}$.
2. **Symmetric Property**: if $i \leftrightarrow j$, then $j \leftrightarrow i$.
3. **Transitive Property**: if $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$.

(15.97)

Property 1 is a consequence of $P_{ij}^{(0)} = \delta_{ij}$; Property 2 is apparent, and 3 is easy to verify (Problem 15.4).

Definition 15.10 (Closed or open set and an irreducible set of states). *A set $\mathcal{C}$ of states is called* ***closed*** *if*

$$P_{ij} = 0 \text{ for all } i \in \mathcal{C}, j \notin \mathcal{C},$$

and is called ***open****, otherwise.*

A set $\mathcal{C}$ is called ***irreducible*** *if*

$$i \leftrightarrow j \text{ for all } i, j \in \mathcal{C}.$$

If all the states $\mathcal{S}$ of a Markov chain are irreducible, the chain is called ***irreducible****.* □

Theorem 15.7 (Property of an irreducible Markov chain). *In an irreducible Markov chain all states belong to the same class: They are all transient, all null-recurrent, or all positive-recurrent. Furthermore, they are either all aperiodic or all periodic with the same period.*

Proof: See p. 447.

Theorem 15.8 (Decomposition of a Markov chain). *If a Markov chain is not an irreducible chain, its state space $\mathcal{S}$ can be uniquely partitioned as*

$$\mathcal{S} = \mathcal{T} \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots ,$$

where $\mathcal{T}$ is the set of transient states, and the $\mathcal{C}_r$'s ($r = 1, 2, \ldots$), are irreducible closed sets of recurrent states.

Proof. See p. 447

15.3.4 Stationary Distribution of an Aperiodic Irreducible Chain

$$\lim_{n\to\infty} \boldsymbol{p}(n) = \boldsymbol{\pi}. \tag{15.98}$$

Then by applying this limit to (15.11),

$$\boxed{\boldsymbol{p}^\top(n+1) = \boldsymbol{p}^\top(n)\boldsymbol{P} \text{ for all } n = 0, 1, 2, \ldots,} \tag{15.11}$$

$$\boldsymbol{\pi}^\top = \boldsymbol{\pi}^\top \boldsymbol{P}. \tag{15.99}$$

Definition 15.11 (Stationary distribution). *A probability distribution $\boldsymbol{\pi}$ satisfying (15.99) is called a **stationary distribution** or an **invariant distribution**: If the initial probability $\boldsymbol{p}(0)$ is set to $\boldsymbol{\pi}$, then $\boldsymbol{p}(n) = \boldsymbol{\pi}$ for all $n \geq 0$.* $\square$

Theorem 15.9 (Stationary distribution of an irreducible aperiodic Markov chain). *In an irreducible aperiodic Markov chain, one of the following two alternatives holds:*

1. *The states are all transient or all null-recurrent; in this case there exists no stationary distribution and*

$$\lim_{n\to\infty} P_{ij}^{(n)} = 0, \quad \textit{for all } i, j \in \mathcal{S}. \tag{15.100}$$

2. *All states are ergodic; in this case there exists a* ***unique stationary distribution*** *and*

$$\lim_{n\to\infty} P_{ij}^{(n)} = \pi_j, \quad \textit{for all } i, j \in \mathcal{S}. \tag{15.101}$$

Furthermore, π_i is equal to the reciprocal of the mean recurrence time for state i, i.e.,

$$\boxed{\pi_i = \frac{1}{\mu_{ii}}, \quad i \in \mathcal{S}.} \tag{15.102}$$

Proof. We skip the proof, since it uses Theorems 15.3 and 15.5 which we skipped. Read pp. 448-449.

The theorem assures us that (15.101) holds whenever the states are ergodic.

$$\lim_{n\to\infty} p_j(n) = \lim_{n\to\infty} \sum_{i\in\mathcal{S}} p_i(0) P_{ij}^{(n)} = \sum_{i\in\mathcal{S}} p_i(0)\pi_j$$
$$= \pi_j, \quad j \in \mathcal{S}. \tag{15.103}$$

Thus, the stationary distribution becomes the **steady-state distribution**.

The stationary distribution can be computed by solving linear equations of (15.99) with the linear constraint condition:

$$\boldsymbol{\pi}^\top \mathbf{1} = 1, \tag{15.104}$$

Alternatively, a simple but useful computational formula can be found

$$\boxed{\boldsymbol{\pi}^\top = \mathbf{1}^\top (\boldsymbol{P} + \boldsymbol{E} - \boldsymbol{I})^{-1}.} \tag{15.107}$$