Lecture 14: Applications of the K-L Expansion and The Poisson Process

ELE 525: Random Processes in Information Systems

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Textbook: Hisashi Kobayashi, Brian L. Mark and William Turin, ***Probability, Random Processes and Statistical Analysis*** (Cambridge University Press, 2012)

Example 13.2: The K-L expansion when noise is white

$$R_N(s,t) = \sigma^2 \delta(s-t). \tag{13.140}$$

The integral equation (13.125) reduces to

$$\sigma^2 \int_0^T \delta(s-t) u(t)\, dt = \lambda u(s), \;\; 0 \le s \le T, \tag{13.141}$$

which is simply $\quad \sigma^2 u(s) = \lambda u(s), \;\; 0 \le s \le T.$

Consider the first orthonormal basis function $u_1(t)$ defined by

$$u_1(t) = \frac{S(t)}{\|S\|}, \;\; \text{where} \;\; \|S\|^2 = \int_0^T |S(t)|^2\, dt. \tag{13.143}$$

The expansion coefficients (c.f (13.113), (13.126)) are

$$x_1 = \frac{1}{\|S\|} \int_0^T S^*(t) X(t)\, dt \qquad s_1 = \frac{1}{\|S\|} \int_0^T S^*(t) S(t)\, dt = \|S\|$$

We could find other orthonormal basis functions $u_k(t)$, k=2, 3, …. But the corresponding expansion coefficients s_k are all zero for k=2, 3, …. Then the test statistic $T(\mathbf{x})$ of (13.132) reduce to

$$T_1(\boldsymbol{x}) = \frac{s_1^* x_1 + s_1 x_1^*}{\sigma^2} = \frac{2\|S\|\Re\{x_1\}}{\sigma^2}. \tag{13.145}$$

$T_1(\boldsymbol{x})$ is a **sufficient statistic**

The solution to the integral equation (13.136) is

$$Q(t) = \frac{S(t)}{\sigma^2} \propto S(t) \tag{13.146}$$

SNR is given by

$$d^2 = \int_0^T Q^*(t)S(t)\,dt = \frac{\|S\|^2}{\sigma^2}. \tag{13.147}$$

Example 13.3: Signal space method for digital communications

$$
\begin{aligned}
H_0 &: X(t) = N(t), \\
H_i &: X(t) = S_i(t) + N(t), \quad i = 1, 2, \ldots, M.
\end{aligned}
\qquad (13.148)
$$

Note that not all signals $S_i(t)$ are linearly independent.

Recall the **Gram-Schmidt orthogonalization** process:

$$u_1(t) \triangleq \frac{S_1(t)}{\|S_1\|},$$

$$r_2(t) = S_2(t) - s_{2,1}u_1(t)$$

$$u_2(t) \triangleq \frac{r_2(t)}{\|r_2\|},$$

$$r_i(t) = S_i(t) - s_{i,1}u_1(t) - s_{i,2}u_2(t) - \cdots - s_{i,i-1}u_{i-1}(t),$$

$$\text{where } s_{i,j} = \int_0^T u_j^*(t) S_i(t)\, dt$$

$$u_i(t) \triangleq \frac{r_i(t)}{\|r_i\|}, \quad i \geq 2.$$

Let us compute the following likelihood ratio, instead of $f(\boldsymbol{x}|H_i$ (Why?)

$$L(\boldsymbol{x}|H_i) \triangleq \frac{f(\boldsymbol{x}|H_i)}{f(\boldsymbol{x}|H_0)} = \exp\left[\sum_{k=1}^{m} \frac{s_{i,k}^* x_k + x_k^* s_{i,k} - |s_{i,k}|^2}{\sigma^2}\right], \quad i = 1, 2, \ldots, M, \qquad (13.149)$$

where

$$x_k = \int_0^T u_k^*(t) X(t)\, dt, \quad k = 1, 2, \ldots, m; \qquad (13.150)$$

$$s_{i,k} = \int_0^T u_k^*(t) S_i(t)\, dt, \quad k = 1, 2, \ldots, m, \text{ and } i = 1, 2, \ldots, M. \qquad (13.151)$$

14.1 The Poisson Process

Definition 14.1 (Poisson process). *A Poisson process of rate λ is a counting process $N(t),\ t \geq 0$, taking values in the set $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$ such that*

(a) $N(0) = 0$, and $N(t)$ is non-decreasing, i.e., if $t_2 > t_1$, then $N(t_2) \geq N(t_1)$.
(b) Its transition probabilities

$$P_{mn}(h) \triangleq P[N(t+h) = n | N(t) = m]$$

are stationary (i.e., independent of t) and satisfy

$$\boxed{P_{mn}(h) = \begin{cases} 1 - \lambda h + o(h), & \text{if } n = m, \\ \lambda h + o(h), & \text{if } n = m + 1, \\ o(h), & \text{if } n \geq m + 2. \end{cases}} \tag{14.2}$$

where $o(h)$ represents a quantity that approaches zero faster than h as $h \to 0$, i.e., $\lim_{h \to 0} \frac{o(h)}{h} = 0$.

(c) If $t > s$, then $N(t) - N(s)$, the number of arrivals in $(s, t]$, is independent of $N(s)$, the number of arrivals in $(0, s]$.

□

Theorem 14.1. *The Poisson counting process $N(t)$ is Poisson distributed with mean λt, i.e.,*

$$P[N(t) = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \quad n = 0, 1, 2, \ldots. \tag{14.3}$$

Proof.

$$p_n(t) \triangleq P[N(t) = n]$$

$N(t+h) = n$ occurs if

(i) $N(t) = n$, and no arrival in $(t, t+h]$

or (ii) $N(t) = n-1$ and one arrival in $(t, t+h]$.

$$p_n(t+h) = (1 - \lambda h + o(h))p_n(t) + (\lambda h + o(h))p_{n-1}(t) + o(h)$$
$$= (1 - \lambda h)p_n(t) + \lambda h p_{n-1}(t) + o(h), \quad n \geq 1,$$

For $n = 0$

$$p_0(t+h) = (1 - \lambda h)p_0(t) + o(h)$$

$$\frac{dp_n(t)}{dt} = \lambda p_{n-1}(t) - \lambda p_n(t), \quad n \geq 1,$$
$$\frac{dp_0(t)}{dt} = -\lambda p_0(t). \qquad p_n(0) = \delta_{n,0} \tag{14.7}$$

(a) Induction method:

From (14.7) we readily find

$$p_0(t) = C_0 e^{-\lambda t}$$

C_0 can be determined as $C_0 = 1$:

$$p_0(t) = e^{-\lambda t}$$

Substituting this result into (14.7) with $n = 1$

$$p_1(t) = \lambda t e^{-\lambda t} + C_1 e^{-\lambda t}$$

where $C_1 = p_1(0) = 0$

$$p_1(t) = \lambda t e^{-\lambda t}$$

We repeat the same procedure

$$p_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \quad n = 0, 1, 2, \ldots . \qquad (14.13)$$

(b) Probability generating function (PGF) method:

$$G(z,t) \triangleq E[z^{N(t)}] = \sum_{n=0}^{\infty} p_n(t) z^n. \tag{14.14}$$

Multiplying (14.7) by z^n, and summing it over n

$$\boxed{\frac{\partial}{\partial t} G(z,t) = \lambda z G(z,t) - \lambda G(z,t) = \lambda(z-1)G(z,t).} \tag{14.15}$$

From the initial (or boundary) condition $p_n(0) = \delta_{n,0}$

$$G(z,0) = 1. \tag{14.16}$$

Then

$$G(z,t) = e^{\lambda(z-1)t} = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} z^n \tag{14.17}$$

By comparing the coefficients of z^n in (14.14) and (14.17), we arrive at (14.13).

Theorem 14.2 (Interarrival times of the Poisson process). *The interarrival times $X_1, X_2, \ldots, X_n, \ldots$ of the Poisson process are independent exponential RVs with mean $1/\lambda$, i.e.,*

$$P[X_n \le x] = 1 - e^{-\lambda x}, \quad x \ge 0, \quad \textit{for all } n. \tag{14.18}$$

Proof. Let us consider first X_1,

$$X_1 > x, \text{ if and only if no arrival in } (0, x]$$

$$\boxed{P[X_1 > x] = P[N(x) = 0] = p_0(x) = e^{-\lambda x}.} \tag{14.19}$$

$$P[X_2 > x | X_1 = t_1] = P[\text{No arrival in } (t_1, t_1 + x] | X_1 = t_1]$$
$$= P[\text{No arrival in } (t_1, t_1 + x]] = e^{-\lambda x}.$$

Similarly,

$$P[X_n > x | X_1 = t_1, X_2 = t_2 - t_1, \ldots, X_{n-1} = t_{n-1} - t_{t-2}]$$
$$= P[\text{No arrival in } (t_{n-1}, t_{n-1} + x]] = e^{-\lambda x}. \tag{14.22}$$

14.1.2 Properties of the Poisson Process

1. Memoryless property of Poisson process

$$R = X - Y,$$

$$P[R \le r | X \ge Y] = 1 - e^{-\lambda r} = F_X(r).$$

2. Superposition of Poisson processes

$$N(t) = \sum_{k=1}^{m} N_k(t)$$

$$G_k(z,t) = E\left[z^{N_k(t)}\right] = e^{-\lambda_k t(1-z)}$$

$$G(z,t) = E\left[z^{\sum_{k=1}^{m} N_k(t)}\right] = \prod_{k=1}^{m} G_k(z,t) = e^{-\lambda t(1-z)},$$

3. Decomposition of a Poisson process

For each arrival, we place it into the kth output substream with the probability r_k

Then, the conditional joint distribution of $N_k(t)$ given $N(t) = n$

$$P[n_1, n_2, \ldots, n_m | n] = \frac{n!}{n_1! n_2! \cdots n_m!} r_1^{n_1} r_2^{n_2} \cdots r_m^{n_m}$$

which is a **multinomial distribution** (cf (3.122) of p. 68).

By multiplying the probability distribution (14.13),

$$P[n_1, n_2, \ldots, n_m] = \frac{n!}{n_1! n_2! \cdots n_m!} r_1^{n_1} r_2^{n_2} \cdots r_m^{n_m} \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$
$$= \prod_{k=1}^{m} \frac{(r_k \lambda t)^{n_k}}{n_k!} e^{-r_k \lambda t} = \prod_{k=1}^{m} P(n_k; r_k \lambda t),$$

where $P(i; a)$ is the Poisson distribution defined by (3.77).

4. Uniformity of Poisson arrivals

$$P[t \leq \tau \leq t + h | \text{an arrival in } (0, T]] = \frac{h}{T} + o(h). \tag{14.31}$$

We can prove (14.31) as follows.

The joint probability that

there are i arrivals in a subinterval $(0, t]$

one arrival in $(t, t + h]$

and $n - i - 1$ arrivals in $(t + h, T]$

$$\left(\frac{(\lambda t)^i}{i!} e^{-\lambda t} \right) \left(\lambda h e^{-\lambda h} \right) \left(\frac{[\lambda(T - t - h)]^{n-i-1}}{(n - i - 1)!} e^{-\lambda(T-t-h)} \right)$$

$$= \frac{\lambda^n h e^{-\lambda T}}{(n-1)!} \binom{n-1}{i} t^i (T - t - h)^{n-i-1}$$

Summing the above expression over the possible values of i,

$$\frac{\lambda^n h e^{-\lambda T}}{(n-1)!} \sum_{i=0}^{n-1} \binom{n-1}{i} t^i (T-t-h)^{n-i-1}$$

$$= \frac{[\lambda(T-h)]^{n-1}}{(n-1)!} \lambda h e^{-\lambda T} \qquad (14.33)$$

Since h is an infinitesimal interval, we rewrite (14.33) as

$$P[n \text{ arrivals in } (0,T] \text{ with an arrival in } (t, t+h]]$$

$$= \frac{(\lambda T)^{n-1}}{(n-1)!} \lambda h e^{-\lambda T} + o(h)$$

$$P[\text{an arrival in } (t, t+h] | n \text{ arrivals in } (0,T]]$$

$$= \frac{\frac{(\lambda T)^{n-1}}{(n-1)!} \lambda h e^{-\lambda T} + o(h)}{\frac{(\lambda T)^n}{n!} e^{-\lambda T}}$$

$$= \frac{nh}{T} + o(h).$$

Since the n arrivals are independent, any one of them will fall into the interval $(t, t+h]$ with equal chance, with probability $\frac{h}{T}$ This final result is independent of n, hence the conditional probability is unconditional. Thus, we have proved (14.31).

5. Infinitesimal Generator of Poisson Process

Let $p_n(t) = P[N(t) = n]$ and

$$\boldsymbol{p}(t) = (p_0(t), p_1(t), \ldots, p_n(t), \ldots)^\top$$

The **infinitesimal generator** or **transition rate matrix**

$$\boldsymbol{Q} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & \cdots \\ 0 & -\lambda & \lambda & 0 & \cdots \\ 0 & 0 & -\lambda & \lambda & \cdots \\ 0 & 0 & 0 & -\lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{14.38}$$

Then, the set of differential-difference equations (14.7) can be compactly represented by

$$\boxed{\frac{d\boldsymbol{p}(t)^\top}{dt} = \boldsymbol{p}(t)^\top \boldsymbol{Q},} \tag{14.39}$$

which is a variant of Kolmogorov's forward equation defined in (16.34).