An Optimum Positioning Receiver for Non-synchronized Mobile Systems

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Abstract — **Taking time-difference-of-arrival (TDOA) is a well-known approach to positioning a mobile station (MS) when the clocks at the base stations (BS) and MS are not synchronized. However, most existing TDOA methods are based on some intuitive geometric arguments, and do not take into account such important system parameters as signal-to-noise ratio (SNR) and effective bandwidth in a cohesive manner. Therefore, they cannot be guaranteed to be optimal in attaining the best achievable positioning accuracy. In this paper, we take systematic steps to obtain an optimum positioning receiver. The problem is formulated from the viewpoint of estimation theory, whereby the relevant system parameters are logically introduced. We first evaluate the best geolocation accuracy for the system, i.e., the corresponding Cramer-Rao Lower Bound (CRLB). We further prove that it can be achieved by the maximum likelihood estimator (MLE) based on TDOA data. This result also implies that the popular least square (LS) based algorithm is not optimal.**

I Introduction

The time-difference-of-arrival (TDOA) based approach has long been accepted as a principle approach to positioning a mobile station (MS) in a non-synchronized mobile communication systems. It simplifies a geolocation algorithm by avoiding estimation of the time-offset between the clocks at the MS and the set of base stations (BS). However, its rationale mainly lies in some triangulation arguments, without integrated consideration of such important system parameters as the signal-to-noise ratio (SNR) and the bandwidth efficiency. Furthermore, a fundamental question remains unanswer: what is the optimum positioning receiver structure?

The theme of the paper is to answer this question with estimation theory [1]. We shall concentrate on a line-of-sight (LOS) scenario to explain the main ideas, since its extension to a non-line-of-sight (NLOS) environment takes no new elements. For the sake of completeness, we discuss it briefly at the end of the text. We first evaluate the theoretical best positioning accuracy, i.e., the Cramer-Rao Lower Bound (CRLB), with the matrix separation formulation developed in our early study [2]. We then show that the maximum likelihood estimates (MLE) of the MS position based on TDOA can asymptotically achieve the CRLB. The *Sherman-Morrison-Woodbury formula* [3] for matrix computation plays an important role in the derivation. The immediate significance of our result suggests that the optimum positioning receiver for non-synchronized system can be decomposed into three steps: first, estimate time delays of signals for each MS and BS pair with the matched filter method; second, obtain TDOA data by arbitrarily selecting one BS as a reference; the last is to find the MLE of the MS position based on TDOAs.

The rest of the paper is structured as follows. We present the problem formulation and compute the CRLB in Section II. In Section III, we obtain the optimum receiver structure. We discuss the NLOS related issues in Section IV. Section V concludes the paper.

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II Problem Formulation

Let us consider a non-synchronized mobile system in a LOS environment. Let $\mathcal{L} = \{1, 2, \cdots, L, L+1\}$ be the set of indices of $L+1$ base stations, which are at $\left\{\underline{p_b} = (x_b, y_b),\ b \in \mathcal{L}\right\}$. The parameter of our interest is certainly the MS position $\underline{p} = (x, y)$. Yet there is an additional unknown parameter, the time offset between clocks at the MS and BS's, l_0/c, where $c = 3 \times 10^8\ m/s$ is the speed of light and l_0 is in the unit of length . Hence we define a 3-dimensional parameter $\underline{\theta} = (\underline{p},\ l_0)$. Let τ_b be the time delay of the received signal at base station b (BS_b), specifically to be

$$\tau_b = \frac{1}{c}\left\{\sqrt{(x_b - x)^2 + (y_b - y)^2} + l_0\right\}. \tag{1}$$

The received signal at BS_b is

$$r_b(t) = A_b s(t - \tau_b) + n_b(t), \quad \text{for } b \in \mathcal{L}, \tag{2}$$

where A_b is the signal amplitude for BS_b, $s(t)$ is the base-band waveform, and $n_b(t)$'s are independent complex-valued white Gaussian noise processes with spectral density $N_0/2$.

The probability density function (p.d.f.) of the observations conditioned on $\underline{\theta}$ is

$$f_{\underline{\theta}}\left(\underline{r}(t)\right) \propto \prod_{b=1}^{L+1} \exp\left\{-\frac{1}{N_0}\int \left|r_b(t) - A_b s(t - \tau_b)\right|^2 dt\right\}. \tag{3}$$

Thus, by casting the NLOS geolocation as a multi-parameter estimation problem, we can obtain the CRLB for the best accuracy of MS position estimate. The key step is to calculate the *Fisher information matrix* respect to $\underline{\theta}$, which is defined as [1]

$$\mathbf{J}_{\underline{\theta}} = E_{\underline{\theta}}\left(\frac{\partial}{\partial\underline{\theta}}\log f_{\underline{\theta}} \cdot \left(\frac{\partial}{\partial\underline{\theta}}\log f_{\underline{\theta}}\right)^T\right), \tag{4}$$

where $\frac{\partial}{\partial\underline{\theta}}\log f_{\underline{\theta}}$ is a 3-dimension column vector and symbol "T" designates complex conjugate and transpose. We employ the matrix separation technique in [2] for the computation task. Since the $f_{\underline{\theta}}(\underline{r})$ in Eq. (3) is a function of τ_b's, which in turn are functions of the parameter $\underline{\theta}$ as in Eq. (1), $\mathbf{J}_{\underline{\theta}}$ can be expressed with *chain rule* as

$$\mathbf{J}_{\underline{\theta}} = \mathbf{H} \cdot \mathbf{J}_{\underline{\tau}} \cdot \mathbf{H}^T, \tag{5}$$

where $\mathbf{H}$ is a $3 \times (L+1)$ matrix, and $\mathbf{J}_{\underline{\tau}}$ is the *Fisher information matrix* respect to $\underline{\tau} = (\tau_0, \tau_1, \cdots, \tau_L)$. We decompose the two matrices as

$$\begin{aligned}\mathbf{H} &= \frac{1}{c}\left(\begin{array}{ccc|c} \cos\phi_1 & \cdots & \cos\phi_L & \cos\phi_{L+1} \\ \sin\phi_1 & \cdots & \sin\phi_L & \sin\phi_{L+1} \\ -- & -- & -- & -- \\ 1 & \cdots & 1 & 1 \end{array}\right) \\ &= \frac{1}{c}\begin{pmatrix} \mathbf{H}_L & \mathbf{h}_{L+1} \\ \mathbf{1}^T & 1 \end{pmatrix},\end{aligned} \tag{6}$$

where angle ϕ_b is determined by

$$\phi_b = \tan^{-1}\frac{y - y_b}{x - x_b},$$

and

$$\begin{aligned}\mathbf{J}_{\underline{\tau}} &= \left(\begin{array}{ccc|c} \lambda_1 & & \mathbf{0} & 0 \\ & \ddots & & \vdots \\ \mathbf{0} & & \lambda_L & 0 \\ -- & -- & -- & -- \\ 0 & \cdots & 0 & \lambda_{L+1} \end{array}\right) \\ &= \begin{pmatrix} \mathbf{\Lambda}_L & \mathbf{0} \\ \mathbf{0} & \lambda_{L+1} \end{pmatrix}.\end{aligned} \tag{7}$$

The entries are $\lambda_b = 8\pi^2\beta^2 R_b$, $b \in \mathcal{L}$, where R_b is the SNR of the received signal at BS_b, i.e.,

$$R_b = \frac{\int |A_b s(t)|^2 dt}{N_0},$$

and the effective bandwidth of the signal waveform, β, is given by

$$\beta^2 = \int f^2 |S(f)|^2 df,$$

with $S(f)$ is the Fourier transform of $s(t)$. BS_{L+1} is selected to be the reference BS for constructing TDOA. The reason for such matrix separation will become clear soon.

Denote $\hat{\underline{\theta}}$ as an estimate of parameters $\underline{\theta}$. Its covariance matrix is $\text{Cov}_{\underline{\theta}}(\hat{\underline{\theta}}) = E_{\underline{\theta}}\left\{(\hat{\underline{\theta}} - \underline{\theta})(\hat{\underline{\theta}} - \underline{\theta})^T\right\}$. The CRLB for $\underline{\theta}$ is then expressed as

$$\text{Cov}_{\underline{\theta}}(\hat{\underline{\theta}}) \geq \mathbf{J}_{\underline{\theta}}^{-1}, \tag{8}$$

where the inequality means that the matrix $(\text{Cov}(\underline{\theta}) - \mathbf{J}_{\underline{\theta}}^{-1})$ is non-negative definite. The inverse of the Fisher information of Eq. (5) can be explicitly written as,

$$\begin{aligned}\mathbf{J}_{\underline{\theta}}^{-1} &= \left(\mathbf{H} \cdot \mathbf{J}_{\underline{\tau}} \cdot \mathbf{H}^T\right)^{-1} \\ &= \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \mathbf{A}^{-1} + \mathbf{F}\mathbf{W}^{-1}\mathbf{F}^T & -\mathbf{F}\mathbf{W}^{-1} \\ -\mathbf{W}^{-1}\mathbf{F}^T & \mathbf{W}^{-1} \end{pmatrix},\end{aligned} \tag{9}$$

where

$$\begin{aligned}&\mathbf{A} = \mathbf{H}_L\mathbf{\Lambda}_L\mathbf{H}_L^T + \lambda_{L+1}\mathbf{h}_{L+1}\mathbf{h}_{L+1}^T, \\ &\mathbf{B} = \mathbf{H}_L\mathbf{\Lambda}_L\mathbf{1} + \lambda_{L+1}\mathbf{h}_{L+1}, \\ &\mathbf{C} = \mathbf{1}^T\mathbf{\Lambda}_L\mathbf{1} + \lambda_{L+1}, \\ &\mathbf{W} = \mathbf{C} - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}, \quad \mathbf{F} = \mathbf{A}^{-1}\mathbf{B},\end{aligned} \tag{10}$$

and the inverses that occur in the expression exist [5].

Since we are primarily interested in the MS position accuracy, we only consider $\left[\mathbf{J}_{\underline{\theta}}^{-1}\right]_{2\times2}$, which is the first 2×2 diagonal matrix of $\mathbf{J}_{\underline{\theta}}^{-1}$, i.e.,

$$\left[\mathbf{J}_{\underline{\theta}}^{-1}\right]_{2\times2} = \mathbf{A}^{-1} + \mathbf{F}\mathbf{W}^{-1}\mathbf{F}^T. \tag{11}$$

III Optimum Positioning Receiver

We now come to the critical proof that $\left[\mathbf{J}_{\underline{\theta}}^{-1}\right]_{2\times2}$ can be attained by the MLE of TDOA measurements. Our strategy is as follows. We first model the TDOA estimates. Its associated CRLB for the positioning accuracy, denoted as $(\mathbf{J})_{TDOA}^{-1}$, is then determined. Since it is the accuracy limit for the MLE method, the last step is to confirm the equivalence relation

$$\left[\mathbf{J}_{\underline{\theta}}^{-1}\right]_{2\times2} = (\mathbf{J})_{TDOA}^{-1}. \tag{12}$$

Consider we estimate the time delay (i.e., TOA) of the received signal of Eq. (2) at a matched filter output. The TOA estimates can be approximated as [2]

$$\rho_b = \tau_b + \eta_b, \quad \text{for } b \in \mathcal{L}, \tag{13}$$

where the measurement error η_b is a Gaussian random variable with $\mathcal{N}(0, \zeta_b^2/2)$, and ζ_b relates to λ_b in Eq. (7) as $2\zeta_b^2 = 1/\lambda_b$. Thus we are able to model the TDOA, produced by taking difference of TOA pairs, as

$$\begin{aligned}\omega_b &= \rho_b - \rho_{L+1} \\ &= (\tau_b - \tau_{L+1}) + (\eta_b - \eta_{L+1}) \\ &= (\tau_b - \tau_{L+1}) + \chi_b, \quad \text{for } b = 1, \cdots, L,\end{aligned} \tag{14}$$

where $\underline{\chi}$ conforms $\mathcal{N}(\underline{0}, \mathbf{\Pi})$, and BS_{L+1} is the reference BS. It is worth pointing out that $\mathbf{\Pi}$ is not a diagonal matrix

$$\begin{aligned}\mathbf{\Pi} &= \begin{pmatrix} \zeta_1^2 + \zeta_{L+1}^2 & \zeta_{L+1}^2 & \cdots & \zeta_{L+1}^2 \\ \zeta_{L+1}^2 & \zeta_2^2 + \zeta_{L+1}^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \zeta_{L+1}^2 \\ \zeta_{L+1}^2 & \cdots & \zeta_{L+1}^2 & \zeta_L^2 + \zeta_{L+1}^2 \end{pmatrix} \\ &= 2\left(\lambda_{L+1}\mathbf{1} \cdot \mathbf{1}^T + \mathbf{\Lambda}_L^{-1}\right),\end{aligned} \tag{15}$$

due to the common reference BS.

In order to compute the CRLB associated with $\{\omega_b\}$, we write its *Fisher information matrix* in the familiar form

$$(\mathbf{J}_{\underline{p}})_{TDOA} = (\mathbf{H})_{TDOA} \cdot (\mathbf{J}_{\underline{\tau}})_{TDOA} \cdot (\mathbf{H})^T_{TDOA}, \tag{16}$$

where

$$(\mathbf{H})_{TDOA} = \mathbf{H}_L - \mathbf{h}_{L+1} \cdot \mathbf{1}^T \tag{17}$$

and $(\mathbf{J}_{\underline{\tau}})_{TDOA} = \mathbf{\Pi}^{-1}$. For further evaluation of $(\mathbf{J}_{\underline{\tau}})_{TDOA}$, we employ *Sherman-Morrison-Woodbury formula* [3], i.e.,

$$\left(\mathbf{D} + \mathbf{U}\mathbf{V}^T\right)^{-1} = \mathbf{D}^{-1} - \mathbf{D}^{-1}\mathbf{U}\left(\mathbf{I} + \mathbf{V}^T\mathbf{D}^{-1}\mathbf{U}\right)^{-1}\mathbf{V}^T\mathbf{D}^{-1}, \tag{18}$$

where $\mathbf{I}$ is an identity matrix, and $\mathbf{D}$, $\mathbf{V}$ and $\mathbf{U}$ are as defined with the appropriate dimensions. Additionally, by noting

$$\begin{aligned} \mathbf{1}^T\mathbf{\Lambda}_L\mathbf{1} &= (1 \;\; 1 \;\; \cdots \;\; 1) \begin{pmatrix} \lambda_1 & & & \mathbf{0} \\ & \lambda_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_L \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \\ &= \sum_{b=1}^{L} \lambda_b, \end{aligned} \tag{19}$$

we obtain

$$\begin{aligned} (\mathbf{J}_{\underline{\tau}})_{TDOA} &= \left(\mathbf{\Lambda}_L^{-1} + \lambda_{L+1}\mathbf{1}\cdot\mathbf{1}^T\right)^{-1} \\ &= \mathbf{\Lambda}_L - \frac{1}{\sum \lambda_b}\mathbf{\Lambda}_L \cdot \mathbf{1} \cdot \mathbf{1}^T \cdot \mathbf{\Lambda}_L. \end{aligned} \tag{20}$$

Submit Eqs. (17) and (20) into Eq. (16), and organize the elements in a proper way as

$$\begin{aligned} &(\mathbf{J}_{\underline{p}})_{TDOA} = \left(\mathbf{H}_L - \mathbf{h}_{L+1}\cdot\mathbf{1}^T\right)\cdot\left(\lambda_0\mathbf{1}\cdot\mathbf{1}^T + \mathbf{\Lambda}_L^{-1}\right)^{-1}\cdot \\ &\left(\mathbf{H}_L - \mathbf{h}_{L+1}\cdot\mathbf{1}^T\right)^T \\ &= \left(\mathbf{H}_L\mathbf{\Lambda}_L\mathbf{H}_L^T + \lambda_{L+1}\mathbf{h}_{L+1}\mathbf{h}_{L+1}^T\right) - \\ &\frac{1}{\sum\lambda_b}\left(\mathbf{H}_L\mathbf{\Lambda}_L\mathbf{1} + \lambda_{L+1}\mathbf{h}_{L+1}\right)\left(\mathbf{H}_L\mathbf{\Lambda}_L\mathbf{1} + \lambda_{L+1}\mathbf{h}_{L+1}\right)^T. \end{aligned} \tag{21}$$

By applying Sherman-Morrison-Woodbury formula again to the inverse of Eq. (21), we arrive at the desired relation right away

$$(\mathbf{J}_{\underline{p}})^{-1}_{TDOA} = \left[\mathbf{J}_{\underline{\theta}}^{-1}\right]_{2\times 2}. \tag{22}$$

The proof can be completed with the fact that the CRLB for TDOA can be achieved the MLE of MS position based on TDOAs, which is not difficult to establish [1].

Before leaving this section, we reiterate some major points. First of all, the optimum positioning in a non-synchronized system can be realized by MLE (or, weighted least square (LS)) of TDOA data. Second, it can serve as a direct evidence that the popular LS based methods are not optimal. Third, the estimation result and its accuracy do not depend on the choice of the reference BS.

IV Comments on NLOS Extension

For the optimum receiver in an NLOS environment, the derivation is exactly parallel to that developed in [2] and [4]. Thus we omit the technical details and present the conclusion only. It contains two parts. If no prior statistical information on NLOS delays is available, we should discard the TOA measurements obtained from the NLOS BS's. We then follow the discussion in the previous section on the LOS receiver. If on the other hand, the prior information is given, the maximum *a posteriori* (MAP) estimator on TDOAs can be shown to be optimal.

V Conclusion

In this paper, we present the practical optimum receiver for a non-synchronized communication system. Its connection to the traditional TDOA approach is clarified. In our future work, we plan to devise a computationally efficient geolocation algorithm, which is based on the receiver structure obtained in this paper.

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