On Achievable Bound for Non-line-of-sight Geolocation *

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ABSTRACT

Non-line-of-sight (NLOS) geolocation becomes an important issue with the fast development of mobile communications in recent years. Several methods have been proposed to address this problem. But we believe that a more comprehensive study on the best geolocation accuracy that these methods may possibly achieve is imperative. In [1], we developed a unified analysis of the Cramer-Rao Lower Bound (CRLB) applicable to NLOS geolocation. However, our further study reported here shows that the CRLB is not achievable in general cases. In this paper, we first present a new result on achievable bounds for NLOS geolocation with time-of-arrival (TOA), maximum likelihood estimation (MLE) and signal strength (SS) based positioning methods. Their physical interpretation is consistent with that of the CRLB and provides an insightful direction towards construction of NLOS geolocation algorithms. We then evaluate the difference between the achievable bound and the CRLB obtained earlier, and verify quantitatively that the CRLB can only be achieved in some specific cases, e.g., when the signal-to-noise ration (SNR) is same for the signals received at each base station. Some numerical results are reported based on simulation experiments.

KEY WORDS

achievable bounds, Cramer-Rao Lower Bound, non-line-of-sight geolocation

1 Introduction

Geolocation in non-line-of-sight (NLOS) environment is an important topic in wireless communications. Several methods [2]–[6] have been proposed to mitigate NLOS effects in geolocation. However, there has been no systematic study reported on what is the best geolocation accuracy that these various methods may possibly achieve, which is of practical and theoretical interest. Yet a complete analysis of NLOS geolocation with multipaths is very complicated and difficult. To make the problem manageable, our current work focuses on the scenario of single (line-of-sight (LOS) or NLOS) propagation path between a base station (BS) and mobile station (MS) pair. We hope that thorough understanding of this simple case will shed light on our investigation into multipath situation. In [1], we made our first attempt in this direction: we presented a unified treatment to obtain the Cramer-Rao Lower Bound (CRLB) for NLOS geolocation with time-of-arrival (TOA), maximum likelihood estimation (MLE), and signal strength (SS) based positioning methods. The CRLB is well-known as a tight lower bound for the variance of an unbiased estimate of some unknown parameter [7].

However, we will show here that the CRLB is not achievable in general. Two themes of this paper are to investigate achievable bounds for NLOS geolocation and its relation with the CRLB from our earlier results. We begin with the simplest case – to find the achievable bound for TOA based positioning in LOS environment. We then demonstrate that the bound for NLOS situation is equivalent to that in which only signals from LOS BS's are to be processed. This is a familiar conclusion that we have encountered in discussing a physical interpretation of the CRLB in the same scenario [1]. It is logically clear the achievable bound should be no less than the CRLB. However, it is not straightforward to evaluate the difference between the two bounds. Hence our next task is to derive and discuss their difference. To complete our discussion we comment briefly on the achievable bounds for the MLE and SS based geolocation, which can be developed in a similar manner as the TOA case.

The rest of the paper is organized as follows. In Section 2, we derive the achievable bound for NLOS geolocation with TOA based methods and relate its physical significance. The difference between the CRLB and achievable bounds is evaluated in Section 3. In Section 4, we make some comments on the achievable bounds for the MLE and SS based geolocation. Section 5 presents some numerical examples. We summarize our conclusions in the last section .

2 Achievable bounds for TOA based geolocation

2.1 LOS case

We first look into the simplest scenario: the achievable bound for TOA based positioning in LOS environment. As we will see soon, it leads naturally to the analysis for NLOS

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situation and maximum likelihood estimation (MLE) and signal strength (SS) based geolocation. The position of an MS of our interest $\underline{p} = (x, y)$ is the parameters to be estimated. Let $\mathcal{L} = \{1, 2, \cdots, L\}$ be the set of indices of $L \geq 3$ base stations, which are located at

$$\{\underline{p_b} = (x_b, y_b),\ b \in \mathcal{L}\}.$$

The time delay estimates are

$$\rho_b = \tau_b + \eta_b, \quad \text{for } b \in \mathcal{L}, \tag{1}$$

with

$$\tau_b = \frac{1}{c}\sqrt{(x_b - x)^2 + (y_b - y)^2}, \tag{2}$$

where the measurement errors $\{\eta_b\}$ are approximated as uncorrelated Gaussian random variables with $\{\mathcal{N}(0, \sigma_b^2)\}$ and $c = 3 \times 10^8\ m/s$ is the speed of light. This model becomes accurate when the TOA's are acquired with a matched filter approach at high SNR (signal-to-noise ratio) [8]. The MS position is to be estimated by processing the data $\{\rho_b,\ b \in \mathcal{L}\}$. From estimation theory, we know the maximum likelihood estimate (MLE) is optimum, and is equivalent to the least square (LS) solution in this model.

It is clear that in a noise free situation, we can obtain the precise (x, y) from Eq. (2). Supposing a small disturbance $\Delta\tau_b$ appears in τ_b and $\tau_b >> \Delta\tau_b$, we have the resulting variation of the MS position estimate by taking difference on both sides of Eq. (2), i.e.,

$$\Delta\tau_b = \frac{1}{c}\cos\theta_b\ \Delta x + \frac{1}{c}\sin\theta_b\ \Delta y, \quad \text{for } b \in \mathcal{L}, \tag{3}$$

where angle ϕ_b is determined by

$$\phi_b = \tan^{-1}\frac{y - y_b}{x - x_b}.$$

Write Eq. (3) in matrix notation as

$$\begin{aligned}\Delta\underline{\tau} &= \begin{pmatrix} \Delta\tau_1 \\ \Delta\tau_2 \\ \vdots \\ \Delta\tau_L \end{pmatrix} = \frac{1}{c}\begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ \cos\theta_2 & \sin\theta_2 \\ \vdots & \vdots \\ \cos\theta_L & \sin\theta_L \end{pmatrix}\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \\ &= \mathbf{H}_2^*\ \Delta\underline{p},\end{aligned} \tag{4}$$

where symbol "*" denotes transpose. Using the subscript "2" in $\mathbf{H}_2$ is to be discriminated from $\mathbf{H}_1$, a similar quantity related NLOS BS's geometric relation to the MS, as we will see in next section.

Consider the physical meaning of the small disturbance $\Delta\tau_b$, we may notice that it can be interpreted as the noise η_b due to imperfect time delay estimation as expressed in Eq. (1). Thus we have $\underline{\eta} = \Delta\underline{\tau}$ for $\tau_b >> \eta_b$ (i.e., at high SNR). Equation (4) then becomes

$$\underline{\eta} = \mathbf{H}_2^*\ \Delta\underline{p}. \tag{5}$$

From Eq. (5), it is straightforward to obtain

$$\Delta\underline{p} = (\mathbf{H}_2\mathbf{H}_2^*)^{-1}\,\mathbf{H}_2\ \underline{\eta}. \tag{6}$$

Note $\Delta\underline{p}$ is essentially the difference between the true MS position $\underline{p}$ and its estimate $\widehat{\underline{p}}$, i.e.,

$$\Delta\underline{p} = \widehat{\underline{p}} - \underline{p}. \tag{7}$$

Thus with Eqs. (6) and (7) the covariance matrix of $\widehat{\underline{p}}$ is

$$\begin{aligned}&E\left\{(\hat{\underline{p}} - \underline{p})(\hat{\underline{p}} - \underline{p})^*\right\} = E\left\{\Delta\underline{p}\cdot\Delta\underline{p}^*\right\} \\ &= (\mathbf{H}_2\mathbf{H}_2^*)^{-1}\,\mathbf{H}_2\cdot E(\underline{\eta}\cdot\underline{\eta}^*)\cdot\mathbf{H}_2\,(\mathbf{H}_2\mathbf{H}_2^*)^{-1} \\ &= (\mathbf{H}_2\mathbf{H}_2^*)^{-1}\,\mathbf{H}_2\cdot\mathbf{\Sigma}_2\cdot\mathbf{H}_2\,(\mathbf{H}_2\mathbf{H}_2^*)^{-1},\end{aligned} \tag{8}$$

where $\mathbf{\Sigma}_2$ is a diagonal matrix of order L as

$$\mathbf{\Sigma}_2 = E(\underline{\eta}\cdot\underline{\eta}^*) = \begin{pmatrix} \sigma_1^2 & & & \mathbf{0} \\ & \sigma_2^2 & & \\ & & \ddots & \\ \mathbf{0} & & & \sigma_L^2 \end{pmatrix}. \tag{9}$$

From the procedure of deriving Eq. (8), the covariance matrix of $\widehat{\underline{p}}$, it is clear that it is for an MLE or LS estimator. Recall that MLE here is optimum among any unbiased solutions. We conclude that Eq. (8) is the best achievable bound for TOA based LOS geolocation.

2.2 NLOS case

We now move to TOA positioning in NLOS environment. Let $\mathcal{B} = \{1, 2, \cdots, B\}$ be the set of indices of all relevant BS's, which are located at $\{\underline{p_b} = (x_b, y_b),\ b \in \mathcal{B}\}$. Let $\mathcal{M} = \{k_1, k_2, \cdots, k_M\}$ be the set of the BS's that receive NLOS signals from the MS. Thus the complement $\mathcal{L} = \mathcal{B} \setminus \mathcal{M}$ is the set of LOS base stations with its cardinality $L = B - M$. We can assume $\mathcal{M} = \{1, 2, \cdots, M\}$ without loss of generality. Two sets of parameters to be estimated are the MS position $\underline{p} = (x, y)$ and NLOS propagation induced path lengths $\underline{l} = (l_1, l_2, \cdots, l_M)$. For notation convenience, we define an $(M+2)$-dimensional vector $\underline{\theta} = (\underline{p},\ \underline{l})$. The time delay estimates are modeled same as Eq. (1), i.e.,

$$\rho_b = \tau_b + \eta_b, \quad \text{for } b \in \mathcal{B},$$

except that

$$\tau_b = \frac{1}{c}\left(\sqrt{(x_b - x)^2 + (y_b - y)^2} + l_b\right), \tag{10}$$

where $l_b = 0$ for $b \in \mathcal{L}$.

With a similar argument as in the previous section, we establish

$$\Delta\underline{\tau} = \underline{\eta} = \mathbf{H}^*\ \Delta\underline{\theta} = \begin{pmatrix} \mathbf{H}_1 & \mathbf{H}_2 \\ \frac{1}{c}\mathbf{I}_M & \mathbf{0} \end{pmatrix}^*\ \Delta\underline{\theta}, \tag{11}$$

where

$$\mathbf{H}_1 = \frac{1}{c}\cdot\begin{pmatrix} \cos\phi_1 & \cos\phi_2 & \cdots & \cos\phi_M \\ \sin\phi_1 & \sin\phi_2 & \cdots & \sin\phi_M \end{pmatrix},$$

$$\mathbf{H}_2 = \frac{1}{c}\cdot\begin{pmatrix} \cos\phi_{M+1} & \cos\phi_{M+2} & \cdots & \cos\phi_B \\ \sin\phi_{M+1} & \sin\phi_{M+2} & \cdots & \sin\phi_B \end{pmatrix},$$

and $\mathbf{I}_M$ is a identity matrix of order M. The covariance matrix of $\widehat{\underline{\theta}}$ therefore becomes

$$E\left\{(\hat{\underline{\theta}}-\underline{\theta})(\hat{\underline{\theta}}-\underline{\theta})^*\right\} = E\left\{\Delta\underline{\theta}\cdot\Delta\underline{\theta}^*\right\}$$
$$= \left(\mathbf{H}\mathbf{H}^*\right)^{-1}\mathbf{H}\cdot E(\underline{\eta}\cdot\underline{\eta}^*)\cdot\mathbf{H}^*\left(\mathbf{H}\mathbf{H}^*\right)^{-1}. \quad (12)$$

$E(\underline{\eta}\cdot\underline{\eta}^*)$ in Eq. (12) can be decomposed into sub-matrices as

$$E(\underline{\eta}\cdot\underline{\eta}^*) = \begin{pmatrix} \mathbf{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma}_2 \end{pmatrix}, \quad (13)$$

where

$$\mathbf{\Sigma}_1 = \begin{pmatrix} \sigma_1^2 & & & \mathbf{0} \\ & \sigma_2^2 & & \\ & & \ddots & \\ \mathbf{0} & & & \sigma_M^2 \end{pmatrix},$$

$$\mathbf{\Sigma}_2 = \begin{pmatrix} \sigma_{M+1}^2 & & & \mathbf{0} \\ & \sigma_{M+2}^2 & & \\ & & \ddots & \\ \mathbf{0} & & & \sigma_B^2 \end{pmatrix}.$$

We may have noticed that $\mathbf{\Sigma}_2$ and $\mathbf{H}_2$ in Eqs. (11) and (13) are essentially same as those defined in the LOS case, which are related to LOS BS's, while $\mathbf{\Sigma}_1$ and $\mathbf{H}_1$ are associated to NLOS BS's.

It takes some calculation to obtain

$$\begin{aligned} & \left(\mathbf{H}\mathbf{H}^*\right)^{-1} \\ = & \begin{pmatrix} \left(\mathbf{H}_2\mathbf{H}_2^*\right)^{-1} & -c\left(\mathbf{H}_2\mathbf{H}_2^*\right)^{-1}\mathbf{H}_1 \\ -c\mathbf{H}_1^*\left(\mathbf{H}_2\mathbf{H}_2^*\right)^{-1} & c^2\mathbf{I}_M + c^2\mathbf{H}_1^*\left(\mathbf{H}_2\mathbf{H}_2^*\right)^{-1}\mathbf{H}_1 \end{pmatrix} \\ = & \begin{pmatrix} \mathbf{G}_1 & \mathbf{G}_2 \\ \mathbf{G}_2^* & \mathbf{C}_1 \end{pmatrix}. \end{aligned} \quad (14)$$

From their definition in Eq. (14), it is clear that

$$\mathbf{G}_1 = \mathbf{G}_1^*, \text{ and } \mathbf{G}_2 = -c\mathbf{G}_1\mathbf{H}_1. \quad (15)$$

With Eqs. (13) and (14), Eq. (12) is rewritten as

$$\begin{aligned} & E\left\{(\hat{\underline{\theta}}-\underline{\theta})(\hat{\underline{\theta}}-\underline{\theta})^*\right\} \quad (16) \\ = & \begin{pmatrix} \mathbf{G}_1 & \mathbf{G}_2 \\ \mathbf{G}_2^* & \mathbf{C}_1 \end{pmatrix}\mathbf{H}\cdot E(\underline{\eta}\cdot\underline{\eta}^*)\cdot\mathbf{H}^*\begin{pmatrix} \mathbf{G}_1 & \mathbf{G}_2 \\ \mathbf{G}_2^* & \mathbf{C}_1 \end{pmatrix}. \end{aligned}$$

However, the quantity of our interest is

$$E\left\{(\hat{\underline{p}}-\underline{p})(\hat{\underline{p}}-\underline{p})^T\right\},$$

the first 2×2 diagonal submatrix of Eq. (16). With Eq. (15), it is evaluated as

$$\begin{aligned} & E\left\{(\hat{\underline{p}}-\underline{p})(\hat{\underline{p}}-\underline{p})^*\right\} \\ & = \begin{pmatrix} \mathbf{G}_1^* & \mathbf{G}_2 \end{pmatrix}\mathbf{H}\cdot\begin{pmatrix} \mathbf{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma}_2 \end{pmatrix}\cdot\mathbf{H}^*\begin{pmatrix} \mathbf{G}_1 \\ \mathbf{G}_2^* \end{pmatrix} \\ & = \mathbf{G}_1^*\mathbf{H}_2\mathbf{\Sigma}_2\mathbf{H}_2^*\mathbf{G}_1^* \\ & = \left(\mathbf{H}_2\mathbf{H}_2^*\right)^{-1}\mathbf{H}_2\cdot\mathbf{\Sigma}_2\cdot\mathbf{H}_2^*\left(\mathbf{H}_2\mathbf{H}_2^*\right)^{-1}. \end{aligned} \quad (17)$$

We may notice that Eq. (17) depends only on LOS BS's and the contribution from NLOS signals are completely ignored, which is very similar as the conclusion we have come up with by studying the Cramer-Rao Lower Bound (CRLB) for NLOS geolocation [1]. Furthermore, comparing Eqs. (8) and (17), we see that

$$\begin{aligned} \mathbf{L} & \equiv E\left\{(\hat{\underline{p}}-\underline{p})(\hat{\underline{p}}-\underline{p})^T\right\} \\ & = \left(\mathbf{H}_2\mathbf{H}_2^*\right)^{-1}\mathbf{H}_2\cdot\mathbf{\Sigma}_2\cdot\mathbf{H}_2^*\left(\mathbf{H}_2\mathbf{H}_2^*\right)^{-1}, \end{aligned} \quad (18)$$

holds for both LOS and NLOS situation.

3 Achievable bounds and CRLB

The Cramer-Rao inequality is well known as a lower bound for variances of any unbiased estimates of some unknown parameters [7]. Let $f_{\underline{\theta}}(\underline{r})$ be the probability density function of observations $\underline{r}$ conditioned on $\underline{\theta}$. The *Fisher information matrix* is given by

$$\mathbf{J}_{\underline{\theta}} = E_{\underline{\theta}}\left\{\frac{\partial}{\partial\underline{\theta}}\log f_{\underline{\theta}}(\underline{r})\cdot\left(\frac{\partial}{\partial\underline{\theta}}\log f_{\underline{\theta}}(\underline{r})\right)^*\right\}, \quad (19)$$

where $E_{\underline{\theta}}\{\cdot\}$ stands for an expected value conditioned on $\underline{\theta}$. The CRLB is then expressed as

$$E_{\underline{\theta}}\left\{(\hat{\underline{\theta}}-\underline{\theta})(\hat{\underline{\theta}}-\underline{\theta})^*\right\} \geq \mathbf{J}_{\underline{\theta}}^{-1}. \quad (20)$$

We have shown in [1] that the CRLB for $E_{\underline{\theta}}\left\{(\hat{\underline{p}}-\underline{p})(\hat{\underline{p}}-\underline{p})^T\right\}$ in NLOS geolocation is expressed as

$$\text{CRLB} = \left(\mathbf{H}_2\cdot\mathbf{\Sigma}_2^{-1}\cdot\mathbf{H}_2^*\right)^{-1}, \quad (21)$$

where $\mathbf{H}_2$ and $\mathbf{\Sigma}_2$ are defined as in Section 2.2 With Eqs. (18), (20) and (21), we should have

$$\begin{aligned} \mathbf{L} & = \left(\mathbf{H}_2\mathbf{H}_2^*\right)^{-1}\mathbf{H}_2\cdot\mathbf{\Sigma}_2\cdot\mathbf{H}_2^*\left(\mathbf{H}_2\mathbf{H}_2^*\right)^{-1} \\ & \geq \left(\mathbf{H}_2\cdot\mathbf{\Sigma}_2^{-1}\cdot\mathbf{H}_2^*\right)^{-1} = \text{CRLB}, \end{aligned} \quad (22)$$

i.e., the achievable bound $\mathbf{L}$ is no less than the CRLB. However, it is not obvious about the gap between $\mathbf{L}$ and CRLB and when the equality is satisfied. Our task now is to establish the quantitative relation between $\mathbf{L}$ and the CRLB.

Take the singular value decomposition (SVD) of $\mathbf{H}_2$ as

$$\mathbf{H}_2 = \mathbf{V}\mathbf{\Sigma}\mathbf{U}_1^*, \quad (23)$$

where $\mathbf{\Sigma}$ is a diagonal matrix of order 2, and $\mathbf{V}$ and $\mathbf{U}_1$ are 2×2 and $L\times 2$ matrices respectively. Recall the number of BS's $L \geq 3$. From matrix theory [9], we understand that the column vectors of $\mathbf{V}$ or $\mathbf{U}_1$ are orthonormal to each other.

Substituting Eq. (23) into Eq. (22), a straightforward computation yields

$$\begin{aligned} & \mathbf{L} - \text{CRLB} \quad (24) \\ & = \mathbf{V}\mathbf{\Sigma}^{-1}\left(\mathbf{U}_1^*\mathbf{\Sigma}_2\mathbf{U}_1 - \left(\mathbf{U}_1^*\mathbf{\Sigma}_2^{-1}\mathbf{U}_1\right)^{-1}\right)\mathbf{\Sigma}^{-1}\mathbf{V}^*. \end{aligned}$$

In order to evaluate the term in the bracket of Eq. (24), we acquire an $L \times (L-2)$ matrix, say $\mathbf{U}_2$, such that $(\mathbf{U}_1 \ \ \mathbf{U}_2)$ is a unitary matrix, i.e,

$$\begin{aligned} &\begin{pmatrix} \mathbf{U}_1 & \mathbf{U}_2 \end{pmatrix}^* \begin{pmatrix} \mathbf{U}_1 & \mathbf{U}_2 \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{U}_1 & \mathbf{U}_2 \end{pmatrix} \begin{pmatrix} \mathbf{U}_1 & \mathbf{U}_2 \end{pmatrix}^* \\ &= \mathbf{I}_L. \end{aligned} \tag{25}$$

Therefore, we have

$$\begin{aligned} &\begin{pmatrix} \mathbf{U}_1^* \\ \mathbf{U}_2^* \end{pmatrix} \mathbf{\Sigma}_2 \begin{pmatrix} \mathbf{U}_1 & \mathbf{U}_2 \end{pmatrix} \\ &= \left[\begin{pmatrix} \mathbf{U}_1^* \\ \mathbf{U}_2^* \end{pmatrix} \mathbf{\Sigma}_2^{-1} \begin{pmatrix} \mathbf{U}_1 & \mathbf{U}_2 \end{pmatrix} \right]^{-1}. \end{aligned} \tag{26}$$

The left hand side of Eq. (26) is

$$\begin{aligned} &\begin{pmatrix} \mathbf{U}_1^* \\ \mathbf{U}_2^* \end{pmatrix} \mathbf{\Sigma}_2 \begin{pmatrix} \mathbf{U}_1 & \mathbf{U}_2 \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{U}_1^*\mathbf{\Sigma}_2\mathbf{U}_1 & \mathbf{U}_1^*\mathbf{\Sigma}_2\mathbf{U}_2 \\ \mathbf{U}_2^*\mathbf{\Sigma}_2\mathbf{U}_1 & \mathbf{U}_2^*\mathbf{\Sigma}_2\mathbf{U}_2 \end{pmatrix}. \end{aligned} \tag{27}$$

Its right hand side is derived as

$$\begin{aligned} &\left[\begin{pmatrix} \mathbf{U}_1^* \\ \mathbf{U}_2^* \end{pmatrix} \mathbf{\Sigma}_2^{-1} \begin{pmatrix} \mathbf{U}_1 & \mathbf{U}_2 \end{pmatrix} \right]^{-1} \\ &= \begin{pmatrix} \mathbf{U}_1^*\mathbf{\Sigma}_2^{-1}\mathbf{U}_1 & \mathbf{U}_1^*\mathbf{\Sigma}_2^{-1}\mathbf{U}_2 \\ \mathbf{U}_2^*\mathbf{\Sigma}_2^{-1}\mathbf{U}_1 & \mathbf{U}_2^*\mathbf{\Sigma}_2^{-1}\mathbf{U}_2 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^* & \mathbf{C} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \mathbf{A}^{-1} + \mathbf{F}\mathbf{W}^{-1}\mathbf{F}^* & -\mathbf{F}\mathbf{W}^{-1} \\ -\mathbf{W}^{-1}\mathbf{F}^* & \mathbf{W}^{-1} \end{pmatrix}, \end{aligned} \tag{28}$$

where

$$\mathbf{W} = \mathbf{C} - \mathbf{B}^*\mathbf{A}^{-1}\mathbf{B}, \ \ \mathbf{F} = \mathbf{A}^{-1}\mathbf{B}, \tag{29}$$

and the inverses that occur in the expressions exist [10].

Substituting Eqs. (27) and (28) into Eq. (26) and noticing $\mathbf{A} = \mathbf{U}_1^*\mathbf{\Sigma}_2^{-1}\mathbf{U}_1$, it is observed that

$$\mathbf{U}_1^*\mathbf{\Sigma}_2\mathbf{U}_1 - \left(\mathbf{U}_1^*\mathbf{\Sigma}_2^{-1}\mathbf{U}_1\right)^{-1} = \mathbf{F}\mathbf{W}^{-1}\mathbf{F}^*. \tag{30}$$

Moreover, from Eq. (29), $\mathbf{W}$ is a symmetric matrix and

$$\begin{aligned} \mathbf{W} &= \mathbf{C} - \mathbf{B}^*\mathbf{A}^{-1}\mathbf{B} \\ &= \mathbf{U}_2^*\mathbf{\Sigma}_2^{-1}\mathbf{U}_2 - \mathbf{U}_2^*\mathbf{\Sigma}_2^{-1}\mathbf{U}_1\left(\mathbf{U}_1^*\mathbf{\Sigma}_2^{-1}\mathbf{U}_1\right)^{-1}\mathbf{U}_1^*\mathbf{\Sigma}_2^{-1}\mathbf{U}_2 \\ &= \mathbf{U}_2^*\left(\mathbf{\Sigma}_2^{-1} - \mathbf{\Sigma}_2^{-1}\mathbf{U}_1\left(\mathbf{U}_1^*\mathbf{\Sigma}_2^{-1}\mathbf{U}_1\right)^{-1}\mathbf{U}_1^*\mathbf{\Sigma}_2^{-1}\right)\mathbf{U}_2 \\ &\geq 0. \end{aligned} \tag{31}$$

The last inequality stands for non-negative definite, which is established using the following result (see [10], pp. 49).

Let $\mathbf{Q}$ be a positive definite $m \times m$ matrix, $\mathbf{P}$ be an $m \times k$ matrix, and $\underline{x}$ be an m-vector. Then

$$\inf \underline{x}\mathbf{Q}\underline{x}^* = \underline{x}^*\mathbf{P}(\mathbf{P}^*\mathbf{Q}^{-1}\mathbf{P})^{-1}\mathbf{P}^*\underline{x}, \tag{32}$$

The inequality in Eq. (31) follows by applying $\mathbf{Q} = \mathbf{\Sigma}_2^{-1}$ and $\mathbf{P} = \mathbf{\Sigma}_2^{-1}\mathbf{U}_1^*$. Since the inverse of $\mathbf{W}$ exists, it is clear that $\mathbf{W} > 0$. Thus

$$\mathbf{F}\mathbf{W}^{-1}\mathbf{F}^* \geq 0, \tag{33}$$

where the equality holds if and only if

$$\mathbf{B} = \mathbf{U}_1^*\mathbf{\Sigma}_2^{-1}\mathbf{U}_2 = 0, \tag{34}$$

where $\mathbf{B}$ has been defined in Eq. (28). An obvious solution to Eq. (34) is

$$\mathbf{\Sigma}_2 = \sigma^2\mathbf{I}_L, \tag{35}$$

i.e., the SNR at each LOS BS's is same, which we refer to as the symmetric situation.

Substituting Eq. (33) into Eq. (24), we establish

$$\mathbf{L} - \text{CRLB} = \mathbf{V}\mathbf{\Sigma}^{-1}\left(\mathbf{F}\mathbf{W}^{-1}\mathbf{F}^*\right)\mathbf{\Sigma}^{-1}\mathbf{V}^* \geq 0, \tag{36}$$

where $\mathbf{F}$ and $\mathbf{W}$ are defined in Eq. (29), and the equality holds if and only if the condition of Eq. (34) is satisfied. Particularly, the CRLB can be achieved for a symmetric situation.

4 Achievable bounds for MLE and SS based geolocation

For MLE and SS based geolocation in NLOS environment [1], a similar procedure can be developed to evaluate the achievable bound and its difference to the CRLB. The conclusion is almost same as the TOA case, except that $\mathbf{\Sigma}_2$ in Eq. (13) now is given by

$$\mathbf{\Sigma}_2 = \begin{pmatrix} \lambda_1^{-1} & & & \mathbf{0} \\ & \lambda_2^{-1} & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_L^{-1} \end{pmatrix}. \tag{37}$$

The diagonal entries differ according to the types of models

$$\lambda_b = \begin{cases} 8\pi^2\beta^2 \cdot R_b, & \text{for MLE,} \\ \frac{2\epsilon^2}{\tau_b^2} \cdot R_b, & \text{for SS,} \end{cases} \tag{38}$$

where R_b is the SNR of the received signal at BS_b, i.e.,

$$R_b = \begin{cases} \frac{\int |A_b s(t)|^2 dt}{N_0} & \text{for MLE,} \\ \frac{\int |\frac{A}{\tau_b^\epsilon} s(t)|^2 dt}{N_0} & \text{for SS,} \end{cases} \tag{39}$$

and ϵ in SS model is the path loss factor [1]. The effective bandwidth of the signal waveform β is given by

$$\beta^2 = \int f^2 |S(f)|^2 df,$$

where $S(f)$ is the Fourier transform of $s(t)$.

5 Numerical examples

We present some numerical examples in this section. Since NLOS BS's do not contribute to enhancing the achievable performance, we assume the BS's considered here receive LOS signals without loss of generality. In simulations, all BS's are distributed evenly along a circle with radius of 4000m. The MS can move freely on the two-dimensional plane and transmit a CDMA signal. The SNR at each BS receiver is 3dB when MS is at (0,0). The CDMA signal from the MS is 5Mcps.

In Figure 1, there are 10 base stations. The achievable bound and the CRLB of MS position accuracy are evaluated with Eqs. (18) and (21) for 3, 4,$\cdots$, BS's out of the ten. The MS is located at (1000m,2500m). As anticipated, the achievable bound are generally greater than the CRLB, and the positioning accuracy improves when more BS's participate in the positioning. We notice from simulations that the gap between the two bounds lies heavily in specific geometric relations among MS and BS's. (It is understood that the SNR also influences the gap, but in a predictable way, according to Eqs. (18) and (21).) From Eq. (35), the gap is zero for a symmetric situation. Figure 2 is to illustrate this point. There are 3 BS's. The MS is on a radius with an angle $\pi/4$ to $x = 0$, moving from (0,0) to $(4000\cos(\pi/4), 4000\sin(\pi/4))$. It shows the gap increases when the geometric configuration are more "asymmetric", although the maximum magnitude of the gap is small, less than 2m in this case.

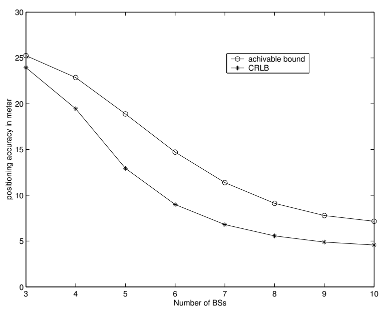

Figure 1. The achievable bound and the CRLB of MS position accuracy vs. the number of BS's involved in geolocating the MS. 10 BS's are located evenly around a circle with radius of 4000m. The SNR is 3dB at each BS receiver when MS is at (0,0).

6 Conclusions

In this paper, we present a closed-form expression of the achievable bound for NLOS geolocation. It is shown that the bound depends only on LOS signals. This observation is consistent with the conclusion we drew earlier for the CRLB [1]. We discuss the quantitative relation between the achievable bound and the CRLB. It is proven that the CRLB is not achievable in most cases, except for the condition of Eq. (34) holds .

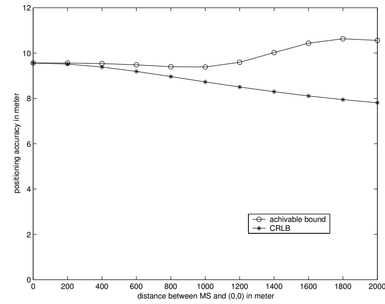

Figure 2. The achievable bound and the CRLB of MS position accuracy vs. the distance between the MS position and (0,0). The MS is on a radius with an angle $\pi/4$ to $x = 0$, moving from (0,0) to $(4000\cos(\pi/4), 4000\sin(\pi/4))$. The SNR is 3dB at each BS receiver when MS is at (0,0).

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