PRINCETON UNIVERSITY
Department of Electrical Engineering
ELE 525: Random Processes in Information Systems
Mid-Term Examination
October 21 (Mon), 2013; 11:00 am-12:20 pm
This is a **Closed Book Exam**.
Assigned as **Homework Assignment #4**, due November 4 (Mon), 2013.

Problem 1: Some Definitions (25 points)
Complete the sentences after "if" in the following statements.

(a) Random variables (RVs) X and Y are said to be **uncorrelated**, if ....

(b) RVs X and Y are said to be **orthogonal**, if ....

(c) RVs X and Y are said to be **independent**, if ....

(d) A sequence of RVs X_n is said to **converge in probability** to RV X, if ....

(e) A sequence of RVs X_n is said to **converge almost surely** to RV X, if ....

Problem 2: A Function of Two Random Variables (25 points)
Given two random variables (RVs) X and Y with joint probability density function (PDF) $f_{XY}(x, y)$, define $Z = X/Y$.

(a) Find the distribution function $F_Z(z)$ and the $f_Z(z)$

(b) Suppose that the joint PDF $f_{XY}(x, y)$ satisfies the following symmetric condition.

$$f_{XY}(-x, -y) = f_{XY}(x, y). \tag{1}$$

Simplify the above expressions for the distribution function and the PDF.

(c) Suppose that X and Y are jointly normally distributed as follows:

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp -\frac{1}{2(1-\rho^2)} \left(\frac{x^2}{\sigma_x^2} - \frac{2\rho xy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2} \right). \tag{2}$$

What is the distribution of the RV Z?

Problem 3: Some Properties of the Laplace Transform (25 points) Consider a real-valued function $f(x), x \geq 0$ which is piecewise continuous and of exponential order, i.e., $|f(x)| \leq Me^{\alpha t}$. We define its Laplace transform by

$$\Phi(s) = \mathcal{L}\{f(x)\} = \int_{0^-}^{\infty} f(x)e^{-sx}\, dx, \tag{3}$$

where $\Phi(s)$ is defined for $\Re(s) > \alpha$.

(a) Show that the Laplace transform of $f'(x) = \frac{d}{dx}f(x)$ is given by

$$\mathcal{L}\{f'(x)\} = s\Phi(s) - f(0^-). \tag{4}$$

(b) Prove the following property known as the *initial value theorem*:

$$\lim_{x\to 0^+} f(x) = \lim_{s\to\infty} s\Phi(s). \tag{5}$$

(c) Prove the *final value theorem*:

$$\lim_{x\to\infty} f(x) = \lim_{s\to 0} s\Phi(s). \tag{6}$$

(d) Consider the PDF of an exponential RV $f(x) = \lambda e^{-\lambda x}u(x)$, where $u(x)$ is the unit step function. Verify that the above three formulas hold for this particular function $f(x)$ and its Laplace transform $\Phi(s)$.

Problem 4: Stationary Process (25 points)

(a) Give the definition of **strict-sense stationarity (SSS)**(also called *strong stationarity*, or simply *stationarity*) of a random process $X(t)$.

(b) Give the definition of **wide-sense stationarity (WSS)** (also called *weak stationarity*, or *2nd-order stationarity*) of a random process $X(t)$.

(c) Show that if $X(t)$ is SSS, it is also WSS.

(d) Suppose that $X(t)$ is a normal (or Gaussian) process. Show that if $X(t)$ is WSS, then it is also SSS.