Lecture 5: Fundamentals of Statistical Analysis and Distributions Derived from Normal Distributions

ELE 525: Random Processes in Information Systems

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September 27, 2013

Textbook: Hisashi Kobayashi, Brian L. Mark and William Turin, ***Probability, Random Processes and Statistical Analysis*** (Cambridge University Press, 2012)

6 Fundamentals of Statistical Data Analysis

6.1 Sample mean and sample variance

The **sample mean** (or the **empirical average**) is defined as

$$\boxed{\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i.} \tag{6.1}$$

Each sample x_i is an **instance** or *realization* of the **associated RV** X_i .
The sample mean of (6.1) is an instance of the **sample mean variable** defined by

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i. \tag{6.2}$$

❖ The expectation is

$$E[\overline{X}] = \mu_X, \tag{6.4}$$

❖ The variance is

$$\boxed{\mathrm{Var}\,[\overline{X}] = \frac{\sigma_X^2}{n}.} \tag{6.8}$$

The sample variance is defined by

$$\boxed{s_x^2 \triangleq \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2.} \tag{6.9}$$

which can be viewed as an instance of the **sample variance variable**

$$S_X^2 \triangleq \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2, \tag{6.10}$$

which is also often called the sample variance. We can show

$$\boxed{E[S_X^2] = \frac{1}{n}\sum_{i=1}^{n}E[Y_i^2] = \sigma_X^2.} \tag{6.12}$$

- ❖ Equations (6.4) and (6.12) show that the *sample mean variable* of (6.2) and the *sample variance variable* (6.10) are **unbiased estimates** of the (population) mean μ_X and the (population) variance σ_X^2, respectively.
- ❖ The square root of the sample variance (6.9), i.e., s_X, is called the **sample standard deviation**.

6.2 Relative frequency and histograms

❖ Consider observed data of sample size n, and they take on k distinct **discrete** values.

Let

n_j = number of times that the jth value is observed, j=1, 2, …, k.

Then

$$\boxed{f_j = \frac{n_j}{n}, \quad j = 1, 2, \ldots k,} \tag{6.13}$$

is called the **relative frequency** of the jth value.

❖ When the underlying RV X is **continuous**, we **group** or **classify** the data.

Divide the range of observations into k **class intervals**, at points c_0, c_1, c_2, …, c_k.

$$\Delta_j \triangleq c_j - c_{j-1}, \ \ j = 1, 2, \ldots, k$$

$$\boxed{h(x) = \frac{f_j}{\Delta_j} = \frac{n_j}{n\Delta_j}\,, \quad \text{for } x \in (c_{j-1}, c_j], \ \ j = 1, 2, \ldots k.} \tag{6.14}$$

is called a **histogram**, and is an estimate of the PDF of the population.

❖ **Cumulative relative frequency**

Let $\{x_{k:} : 1 \le k \le n\}$ be n observations in the order observed, and

$\{x_{(i):} : 1 \le i \le n\}$ be the same observations in order of magnitude.

$H(x)$ be the frequency of observations that are smaller than or equal to x :

$$H(x) = \begin{cases} 0, \text{ for } x < x_{(1)} \\ \frac{i}{n}, \text{ for } x_{(i)} \le x < x_{(i+1)}, \quad i = 1, 2, \ldots, n-1 \\ 1, \text{ for } x \ge x_{(n)}, \end{cases} \qquad (6.15)$$

which can be more concisely written as

$$H(x) = \frac{1}{n} \sum_{i=1}^{n} u(x - x_{(i)}) = \frac{1}{n} \sum_{k=1}^{n} u(x - x_k), \quad -\infty < x < \infty. \qquad (6.17)$$

- ❖ When **grouped data** are presented as a cumulative relative frequency distribution, it is called the **cumulative histogram**.
- ❖ The cumulative histogram is far less sensitive to variation in class lengths than the histogram.

6.3 Graphical presentations

6.3.1 Histogram on probability paper

6.3.1.1 Testing the normal distribution hypothesis

For a given distribution function *F*(*x*), let

$$P = F(x) \tag{6.18}$$

The inverse

$$x_P = F^{-1}(P) \tag{6.19}$$

is the value of *x* that corresponds to the cumulative probability *P*.

- x_P is called the *P*-**fractile** (or P-percentile or P-quantile).

- Consider the **standard normal distribution**

$$\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} \exp\left(-\frac{t^2}{2}\right) dt. \tag{6.20}$$

The fractile u_P of the distribution *N*(0,1) is

$$u_P = \Phi^{-1}(P). \tag{6.21}$$

For a given cumulative relative frequency $H(x)$, we wish to test whether

$$H(x) \cong \Phi\left(\frac{x-\mu}{\sigma}\right) \tag{6.22}$$

holds for some μ and σ. Testing the above is equivalent to testing the relation

$$u_{H(x)} \approx \frac{x-\mu}{\sigma} \tag{6.23}$$

The plot of $u_{H(x)}$ versus x forms a step (or staircase) curve:

$$\boxed{u_{H(x)} = \begin{cases} -\infty, \text{ for } x < x_{(1)} \\ u_{i/n}, \text{ for } x_{(i)} \leq x < x_{(i+1)}, \quad i = 1, 2, \ldots, n-1, \\ \infty, \quad \text{for } x \geq x_{(n)}. \end{cases}} \tag{6.24}$$

The plot in the (x, u)-coordinates of the staircase function

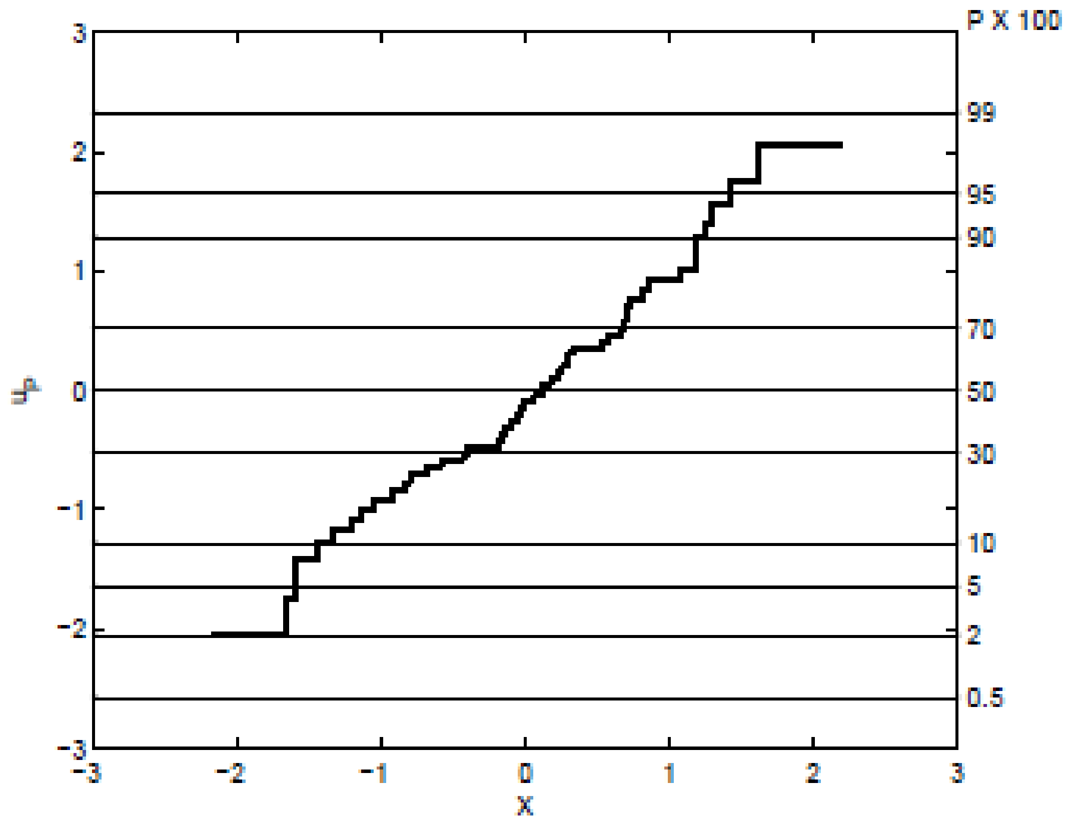
$$u = u_{H(x)} \tag{6.25}$$

is called the **fractile diagram**, and provides an estimate of the straight line

$$u = \frac{x-\mu}{\sigma} \tag{6.26}$$

❖ The **probability paper**

On the ordinate axis, the values $P=\Phi(u)$ are marked, rather than the u values.



(a)

Figure 6.1 The fractile diagram of normal variates: (a) step curve; (n=50)

❖ The **dot diagram**: Instead of the step curve, we plot n points $(x_{(i)}, (i-½)/n)$, which are situated at the middle points.

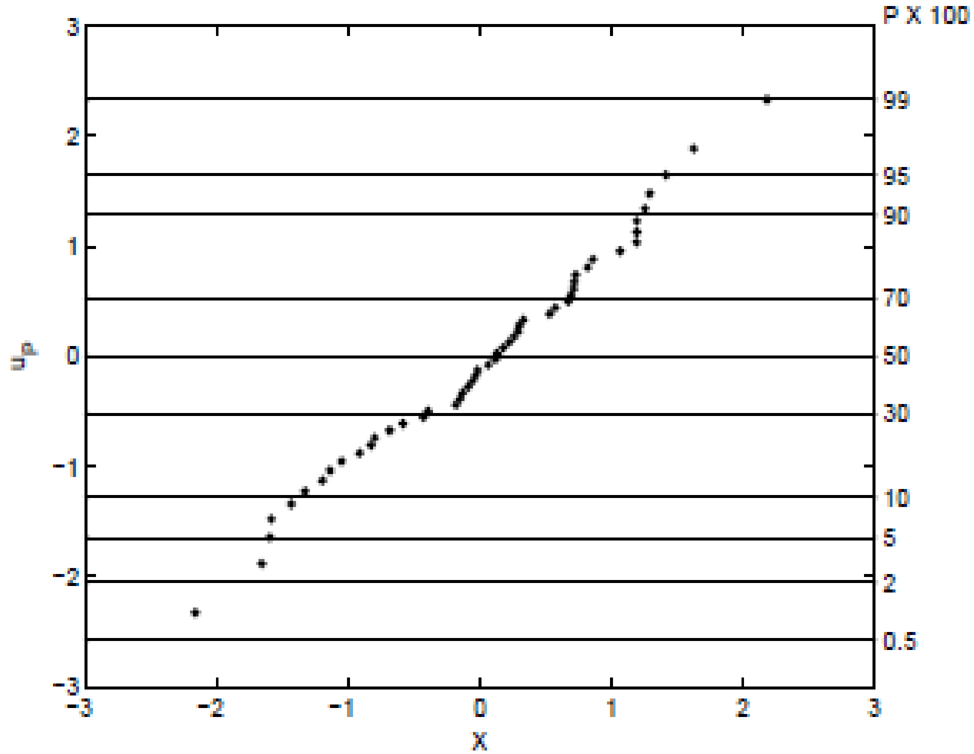


Figure 6.1 The fractile diagram of normal variates: (b) dot diagram. (n=50)

6.3.1.2 Testing the log-normal distribution hypothesis

The **log-normal paper**: Modify the probability paper by changing the horizontal axis from the linear scale to the logarithmic scale, i.e., $\log_{10} x$.

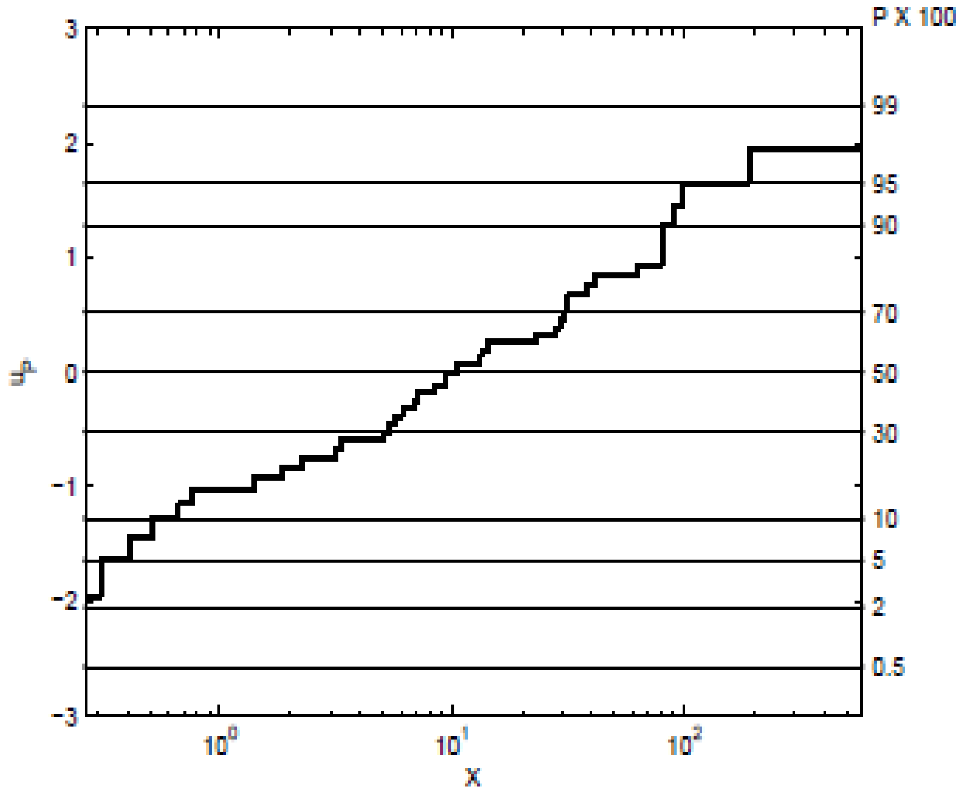


Figure 6.2 The fractile diagram of log-normal variates: (a) step curve

n=50, x_i= exp y_i where y_i is drawn from $N(2,4)$.

6.3.2 Log-survivor function curve

- The **survivor function** or the **survival function**:

$$S_X(t) \triangleq P[X > t] = 1 - F_X(t) \tag{6.27}$$

- The **log-survivor function** or the **log survival function**:

$$\boxed{\log S_X(t) = \log\ (1 - F_X(t)).} \tag{6.28}$$

- The **sample log-survivor function** or **empirical log-survivor function**:

$$\boxed{\log\ [1 - H(t)],} \tag{6.31}$$

where H(x) is the *cumulative relative frequency* (for ungrouped data) or the *cumulative histogram* (for grouped data).

- ✓ For the ungrouped case: Plot

$$\log\left(1 - \frac{i}{n}\right), \quad 1 \le i \le n \tag{6.32}$$

against $x_{(i)}$

In order to avoid difficulty at i=n, we may modify (6.32) into

$$\log\left(1-\frac{i}{n+1}\right), \quad 1 \le i \le n. \tag{6.33}$$

Example: A mixed exponential (or hyperexponential) distribution:

$$F_X(x) = \pi_1(1-e^{-\alpha_1 x}) + \pi_2(1-e^{-\alpha_2 x}), \quad \alpha_1 > \alpha_2, \quad \pi_1 + \pi_2 = 1, \tag{6.29}$$

$$\log S_X(t) = \log\left[\pi_1 e^{-\alpha_1 t} + \pi_2 e^{-\alpha_2 t}\right]$$
$$\approx \begin{cases} -\alpha_1 t + \log \pi_1, \text{ for small } t \\ -\alpha_2 t + \log \pi_2, \text{ for large } t. \end{cases} \tag{6.30}$$

Numerical example:

π_2=0.0526, π_1 =1-π_2, α_2 = 0.1 and α_1 = 2.0

Note: To be consistent with the assumption in (6.29), we should exchangd the subscripts 1 and 2.

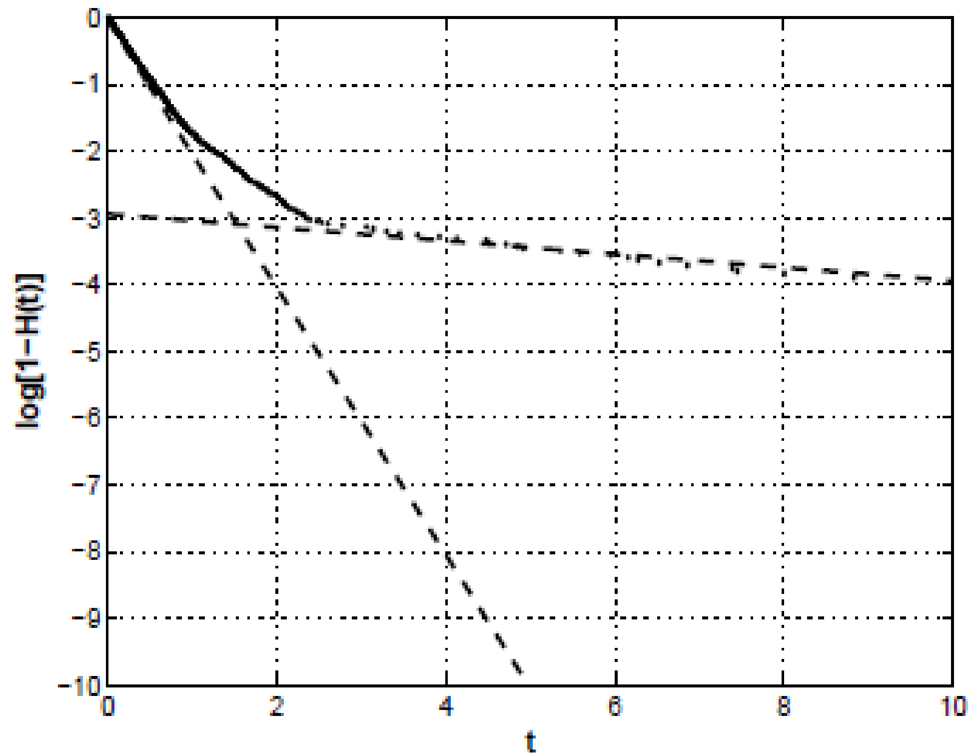


Figure 6.3 The log-survivor function of a mixed-exponential (or hyper-exponential) distribution with $\pi_1 = 0.0526$, $\pi_2 = 1 - \pi_1$, $\alpha_1 = 0.1$, and $\alpha_2 = 2.0$.

Correction to the figure caption: Exchange the subscripts 1 and 2 of π and α to be consistent with (6.29) and (6.30)

6.3.3 Hazard function and mean residual life curves

❖ The **hazard function** or the *failure rate*:

$$\boxed{h_X(t) = \frac{f_X(t)}{S_X(t)} = \frac{f_X(t)}{1 - F_X(t)},} \tag{6.36}$$

which is called the **completion rate function,** when *X* represents a service time variable.

❖ The survivor function and the hazard function are related by

$$\boxed{S_X(x) = e^{-\int_0^x h_X(t)\,dt}, \quad x \geq 0,} \tag{6.38}$$

and

$$h_X(t) = -\frac{d \log S_X(t)}{dt}, \quad t \geq 0. \tag{6.39}$$

❖ Given that the service time variable *X* is greater than *t*,

$$R = X - t \tag{6.40}$$

is the residual life conditioned on X > t.

The mean residual life function

$$\boxed{R_X(t) = E[R|X > t] = \frac{\int_t^\infty S_X(u)\,du}{S_X(t)}.} \tag{6.41}$$

$$R_X(0) = \int_0^\infty S_X(u)\,du = E[X], \tag{6.42}$$

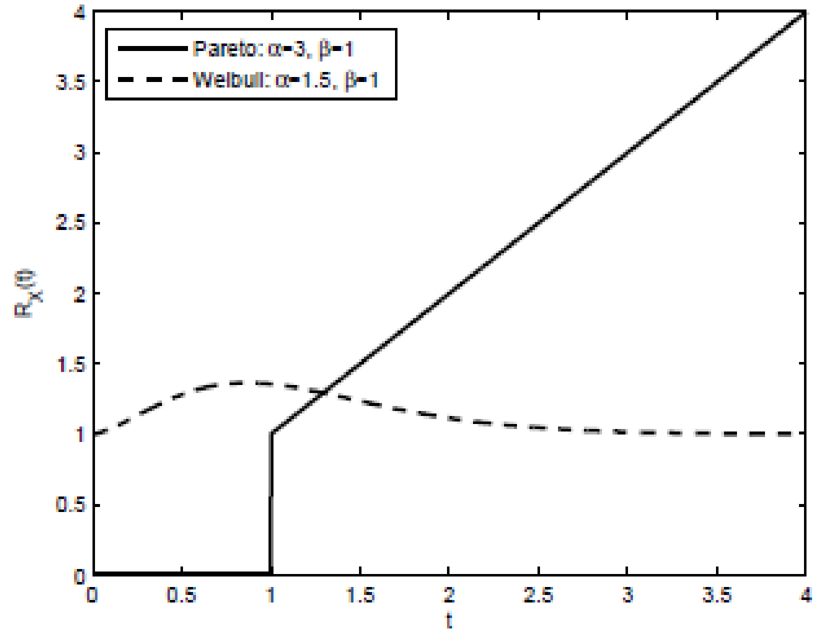


Figure 6.5 The mean residual life curves of a Pareto distribution with $\alpha = 3.0$ and $\beta = 1.0$, and a Weibull distribution with $\alpha = 1.5$ and $\beta = 1.0$.

6.3.4 Dot diagram and correlation coefficient

Dot or **scatter diagram**:

$$(x_i, y_i); 1 \leq i \leq n \tag{6.43}$$

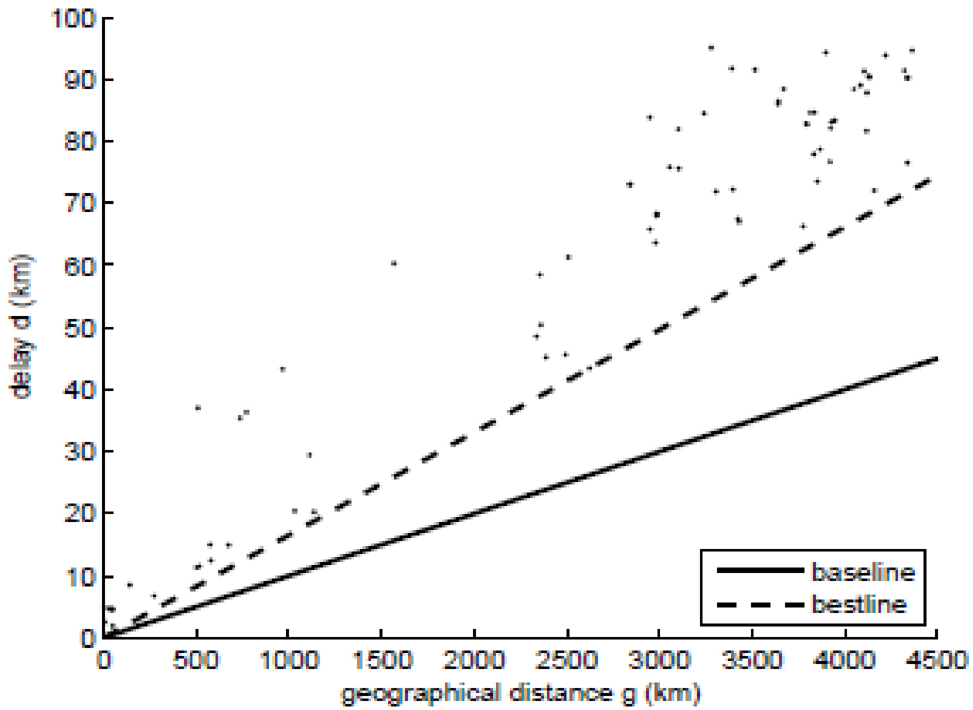


Figure 6.6 Scatter diagram of delay measurements from Internet host at Stanford University to 79 other hosts across the U.S. [363]

Correlation coefficient

$$\boxed{\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}.} \tag{6.46}$$

where

$$\sigma_{XY} \triangleq \mathrm{Cov}[X, Y] = E[(X - \mu_X)(Y - \mu_Y)] = E[XY] - \mu_X \mu_Y. \tag{6.45}$$

X and Y are said to be **properly linearly correlated** if

$$P[aX - bY = c] = 1. \tag{6.48}$$

$$\mathrm{Var}\,[aX - bY - c] = 0, \tag{6.49}$$

$$\rho_{XY} = +1 \quad \text{or} \quad -1 \tag{6.50}$$

depending on whether *ab* is positive or negative.

Conversely, if ρ= ± 1, then (Problem 6.17)

$$P\left[\mp\frac{(X-\mu_X)}{\sigma_X}+\frac{Y-\mu_Y}{\sigma_Y}=0\right]=1. \tag{6.51}$$

❖ The **sample variance** based on observations $\{(x_i, y_i): 1 \le i \le n\}$

$$\boxed{\begin{aligned} s_{xy} &= \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\overline{x})(y_i-\overline{y}) \\ &= \frac{1}{n-1}\sum_{i=1}^{n}x_i y_i - \frac{n\overline{xy}}{n-1}, \end{aligned}} \tag{6.52}$$

❖ The sample correlation coefficient

$$\boxed{r_{xy}=\frac{s_{xy}}{s_x s_y},} \tag{6.53}$$

7 Distributions Derived from the Normal Distribution

7.1 Chi-Squared Distribution

Let U_i, $1 \le i \le n$ be n i.i.d RVs with the standard normal distribution $N(0,1)$.

Define

$$\chi_n^2 = \sum_{i=1}^{n} U_i^2. \tag{7.1}$$

The PDF of this RV (Problem 7.2)

$$f_{\chi_n^2}(x) = \frac{x^{(n/2)-1}e^{-x/2}}{2^{n/2}\Gamma\left(\frac{n}{2}\right)}dx, \quad 0 \le x < \infty, \tag{7.2}$$

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt. \tag{7.3}$$

$$\Gamma(1) = 1, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{7.4}$$

$$\Gamma\left(\frac{n}{2}\right) = \begin{cases} \left(\frac{n}{2}-1\right)!, & \text{for } n \text{ even} \\ \left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}, & \text{for } n \text{ odd.} \end{cases} \tag{7.5}$$

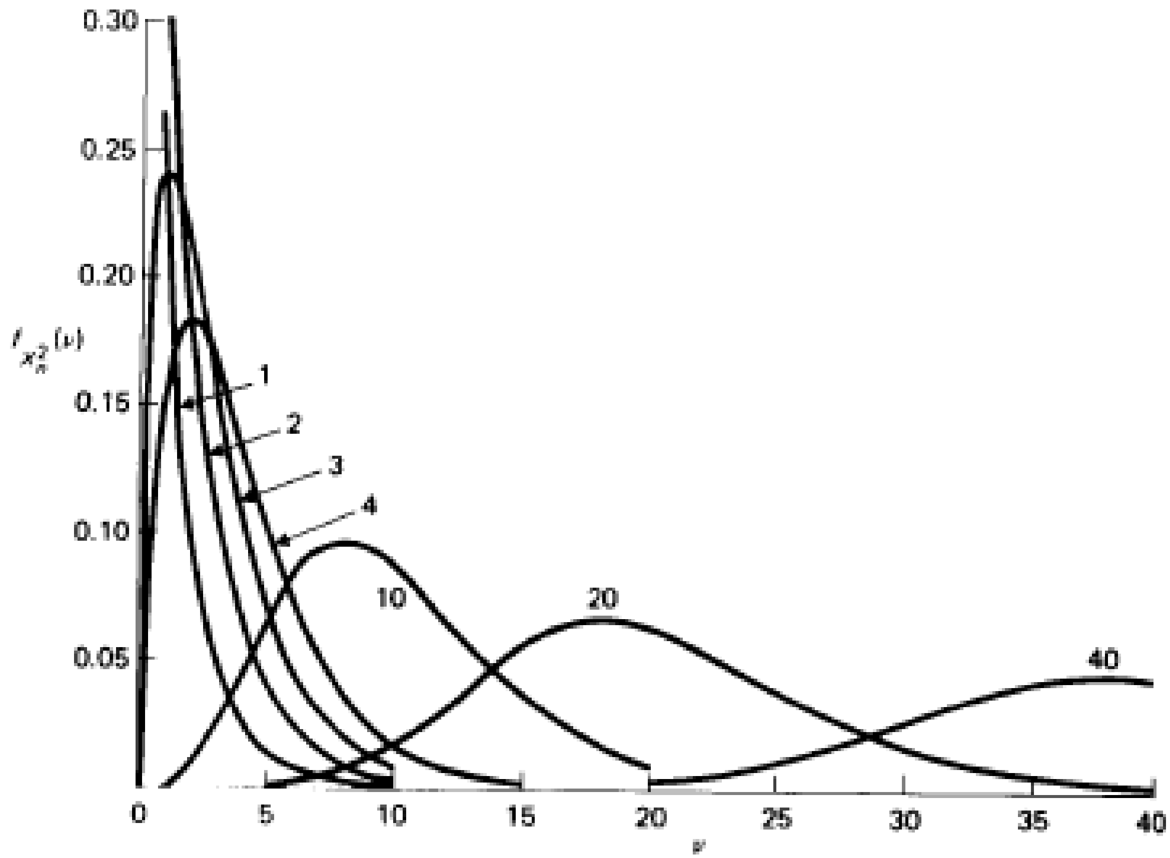


Figure 7.1 The χ_n^2 distribution with degree of freedom n.

n=1 $$f_{\chi_1^2}(x) = \frac{x^{-1/2}e^{-x/2}}{\sqrt{2\pi}}, \quad x > 0. \tag{7.6}$$

n=2 $$f_{\chi_2^2}(x) = \frac{e^{-x/2}}{2}, \quad x \geq 0 \tag{7.7}$$

n=3 $$f_{\chi_3^2}(x) = \frac{x^{1/2}e^{-x/2}}{\sqrt{2\pi}}, \quad x \geq 0. \tag{7.8}$$

$$E[\chi_n^2] = n, \tag{7.9}$$

$$\mathrm{Var}[\chi_n^2] = 2n. \tag{7.10}$$

Mode: n-2

The relations to other distributions:

❖ Let

$$Y_n = \frac{\chi_n^2}{2} \tag{7.11}$$

$$f_{Y_n}(y) = \frac{y^{(n/2)-1}e^{-y}}{\Gamma\left(\frac{n}{2}\right)}, \tag{7.12}$$

which is a special case λ =1, β =*n*/2 in the **gamma distribution** (4.30)

$$f(y) = \begin{cases} \frac{e^{-\lambda y}}{\Gamma(\beta)}\lambda(\lambda y)^{\beta-1}, & y \geq 0 \\ 0, & y < 0, \end{cases} \tag{7.13}$$

❖ The case where *n* is an even integer:

$$n = 2k, \tag{7.15}$$

$$f_{Y_{2k}}(y) = \frac{y^{k-1}e^{-y}}{(k-1)!}, \tag{7.16}$$

which is the *k*-stage **Erlang distribution** with mean *k*.

❖ The relation to the **Poisson distribution**

$$P[\chi^2_{2k} > 2\lambda] = \int_\lambda^\infty \frac{y^{k-1}e^{-y}}{(k-1)!}dy$$
$$= \int_\lambda^\infty P(k-1;y)dy = Q(k-1;\lambda), \quad (7.17)$$

Example 7.1: Independent observations from $N(\mu, \sigma^2)$

❖ Case 1: An estimate of σ^2, when the population mean μ is known

$$\tilde{s}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2. \quad (7.18)$$

$$\tilde{s}^2 = \frac{\sigma^2}{n}\sum_{i=1}^{n}U_i^2, \quad (7.19)$$

where

$$U_i = \frac{X_i - \mu}{\sigma}, \quad 1 \le i \le n, \quad (7.20)$$

Thus, we can write

$$\tilde{s}^2 = \frac{\sigma^2}{n}\chi^2_n. \quad (7.21)$$

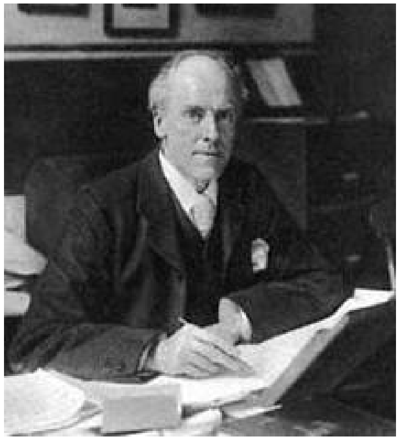
❖ An estimate of σ^2 when μ is unknown.

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2. \tag{7.22}$$

We can show (Problem 7.1)

$$s^2 = \frac{\sigma^2}{n-1}\chi^2_{n-1}. \tag{7.26}$$

Karl Pearson (1857-1936) was a British statistician who applied statistics to biological problems of heredity and evolution



7.2 Student's t-Distribution

The sample mean $\bar{X}$ of n independent observations $\{X_1, X_2, \ldots, X_n\}$ from $N(\mu, \sigma^2)$ is normally distributed according to $N(\mu, \sigma^2/n)$.

Thus,

$$U = \frac{(\bar{X} - \mu)\sqrt{n}}{\sigma} \tag{7.30}$$

is a standard normal variable.

We wish to estimate the population mean μ .

- ❖ If σ is known, we can use the table of the standard normal distribution to test whether U is significantly different from 0.

- ❖ If σ is unknown, we use

$$t_{n-1} = \frac{(\bar{X} - \mu)\sqrt{n}}{s}. \tag{7.31}$$

Using (7.26)

$$t_{n-1} = \frac{(\bar{X} - \mu)\sqrt{n}/\sigma}{s/\sigma} = \frac{U}{\sqrt{\chi^2_{n-1}/(n-1)}}. \tag{7.32}$$

The distribution of the variable t_k is called the (**Student's**) ***t*-distribution** with k degrees of freedom (d.f.).
Its PDF is given by (Problem 7.6)

$$f_{t_k}(t) = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\sqrt{\pi k}}\left(1+\frac{t^2}{k}\right)^{-(k+1)/2}, \quad -\infty < t < \infty. \tag{7.33}$$

k=1,

$$f_{t_1}(t) = \frac{1}{\pi(1+t^2)}, \tag{7.34}$$

which is called the **Cauchy's distribution**.

k=2,

$$f_{t_2}(t) = (2+t^2)^{-3/2}, \tag{7.35}$$

which has zero mean but infinite variance.

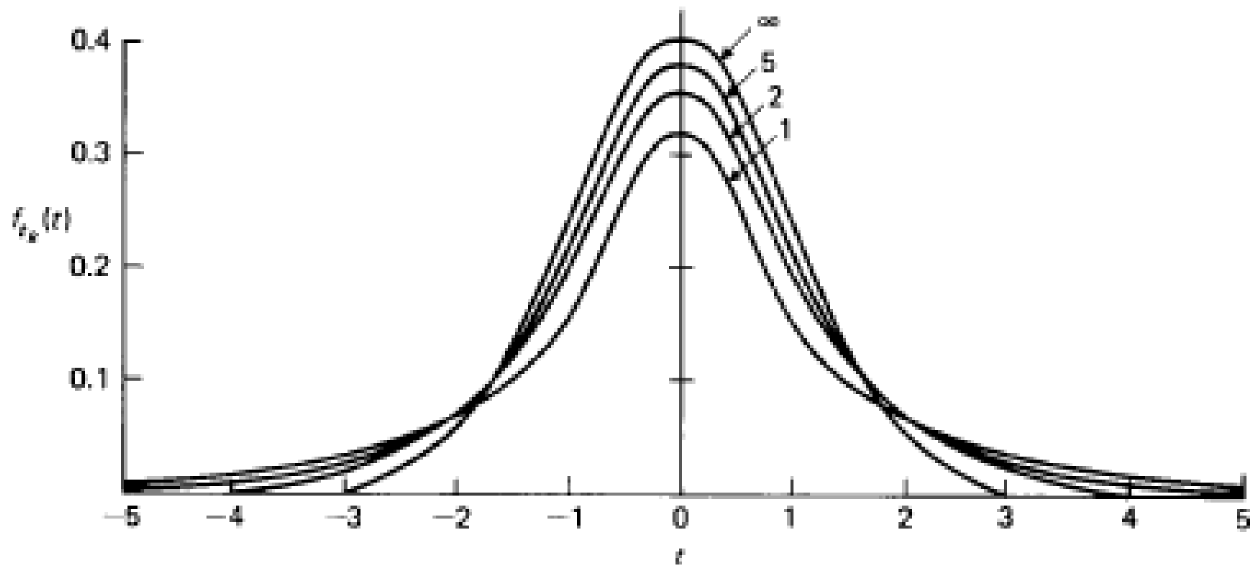


Figure 7.2 The student's t-distribution with k degrees of freedom $(k = 1, 2, 5, \infty)$.

$$E[t_k^{2r}] = E[(\chi_1^2)^r]E\left[\left(\frac{\chi_k^2}{k}\right)^{-r}\right], \tag{7.36}$$

$$E[t_k^{2r}] = \frac{k^r\Gamma\left(\frac{1}{2}+r\right)\Gamma\left(\frac{k}{2}-r\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{k}{2}\right)}, \tag{7.37}$$

$$E[t_k] = 0, \qquad \mathrm{Var}\,[t_k] = \frac{k}{k-2}. \tag{7.38}$$

William S. Gosset (1876-1937) was a statistician of the Guinness brewing company.

7.3 Fisher's *F*-distribution

RVs V_1 and V_2 are independent and are χ^2 distributed with n_1 and n_2 degrees of freedom (d.f.), respectively. Then the variable *F* defined by

$$F = \frac{V_1/n_1}{V_2/n_2} . \qquad (7.39)$$

has the following PDF:

$$f_F(x) = \frac{\Gamma\left(\frac{n_1+n_2}{2}\right)\left(\frac{n_1}{n_2}\right)^{n_1/2}}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} x^{(n_1/2)-1}\left(1+\frac{n_1 x}{n_2}\right)^{-(n_1+n_2)/2} \qquad (7.40)$$

which is called the **F-distribution with (n_1 , n_2)**, also called the **Snedecor distribution**.

$$E[F^r] = \frac{\left(\frac{n_2}{n_1}\right)^r \Gamma\left(\frac{n_1}{2}+r\right)\Gamma\left(\frac{n_2}{2}-r\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \qquad (7.41)$$

which exists for $-n_1 < 2r < n_2$.

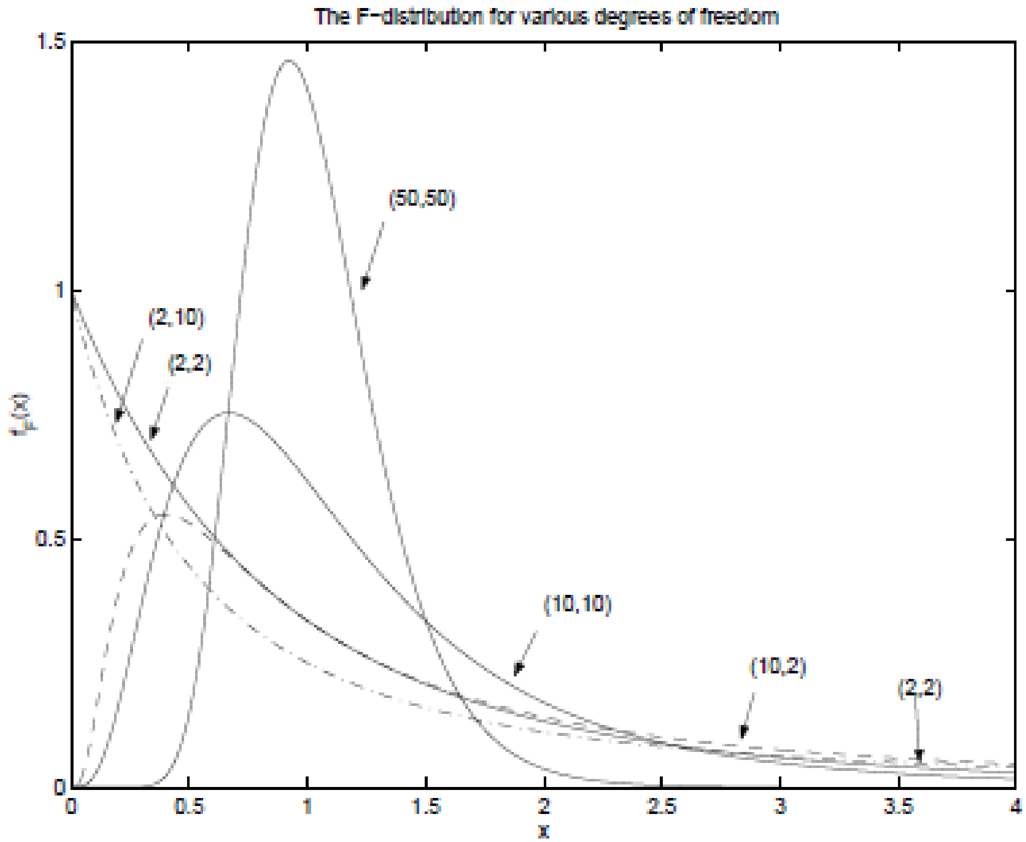


Figure 7.3 The F-distributions for various degrees of freedom (n_1, n_2).

$$E[F] = \frac{n_2}{n_2 - 2} \quad \text{for} \quad n_2 > 2 \tag{7.42}$$

$$\text{Var}\,[F] = \frac{2n_2^2(n_1 + n_2 - 2)}{n_1(n_2 - 2)^2(n_2 - 4)} \quad \text{for} \quad n_2 > 4. \tag{7.43}$$

$$\text{mode}\, F = \frac{n_2(n_1 - 2)}{n_1(n_2 + 1)}. \tag{7.44}$$

7.4 Log-normal distribution

A *positive* RV *X* is said to have the **log-normal distribution** if

$$Y = \ln X$$

Is normally distributed, i.e.,

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left\{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right\}, \quad -\infty < y < \infty. \tag{7.46}$$

Then, by using $dy = \frac{dx}{x}$, and $f_Y(y)\,dy = f_X(x)\,dx$, we readily find

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_Y x} \exp\left\{-\frac{(\ln x-\mu_Y)^2}{2\sigma_Y^2}\right\}, \quad x > 0. \tag{7.47}$$

In order to find the expectation and variance , we use the **moment generating function (MGF)** (to be studied in Section 8.1)

$$M_Y(t) = E[e^{tY}] = \exp\left\{\mu_Y t + \frac{\sigma_Y^2 t^2}{2}\right\}. \tag{7.48}$$

$$\mu_X = E[X] = E[e^Y] = M_Y(1) = \exp\left\{\mu_Y + \frac{\sigma_Y^2}{2}\right\}. \tag{7.49}$$

$$E[X^2] = E[e^{2Y}] = M_Y(2) = \exp\left\{2\mu_Y + 2\sigma_Y^2\right\} = \mu_X^2 e^{\sigma_Y^2}. \tag{7.50}$$

Then

$$\sigma_X^2 = \mu_X^2 \left(\exp\left\{ \sigma_Y^2 \right\} - 1 \right). \qquad (7.51)$$

From (7.49) and (7.51) we find

$$\mu_Y = \ln \mu_X - \frac{1}{2} \ln \left(1 + \frac{\sigma_X^2}{\mu_X^2} \right), \qquad (7.52)$$

$$\sigma_Y^2 = \ln \left(1 + \frac{\sigma_X^2}{\mu_X^2} \right). \qquad (7.53)$$