Lecture 21:

Wiener Filter Theory: Part 2

ELE 525: Random Processes in Information Systems

Hisashi Kobayashi

Department of Electrical Engineering
Princeton University
December 4, 2013

Textbook: Hisashi Kobayashi, Brian L. Mark and William Turin, ***Probability, Random Processes and Statistical Analysis*** (Cambridge University Press, 2012)

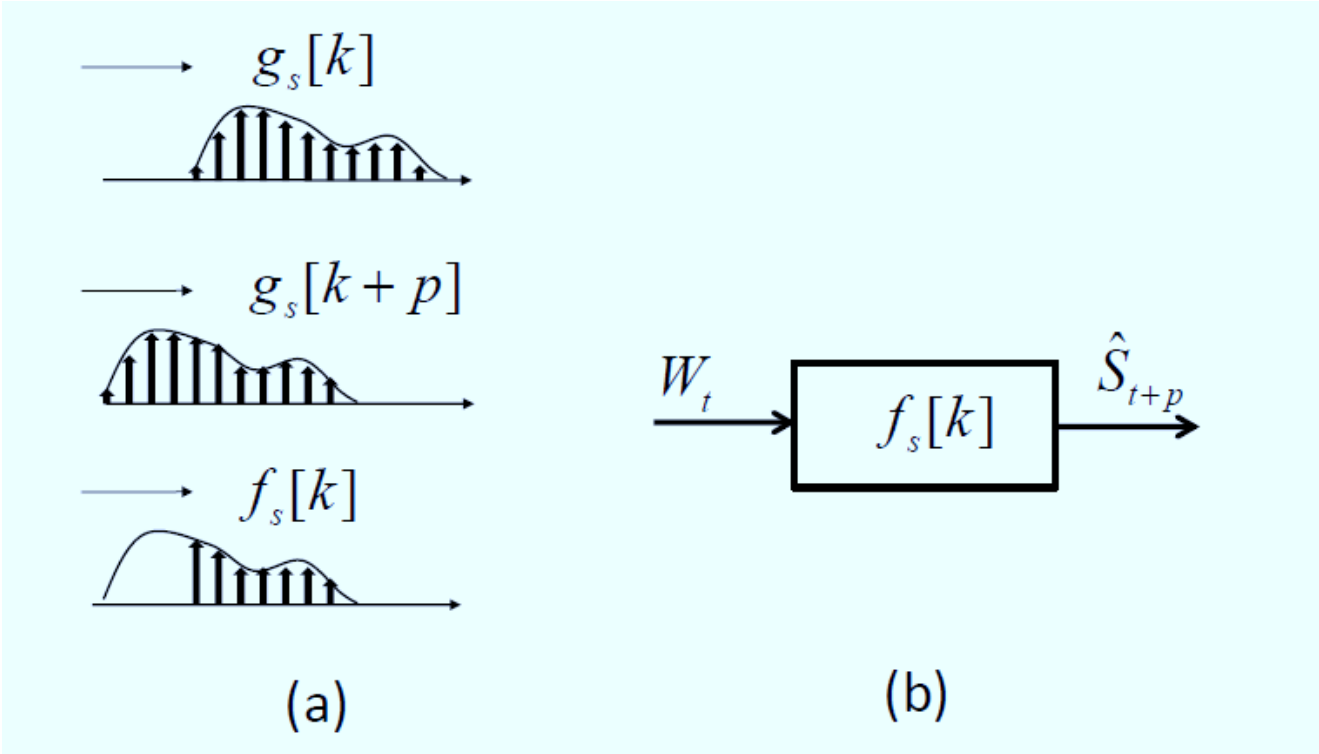


Figure 22.6 (a) The relation between $g_s[k]$ and $f_s[k]$; (b) An optimal predictor $f_s[k] = g_s[k+p]$ that acts on white noise.

Since the white noise W_t can be created by passing the signal S_t into a filter $1/G_s(z)$

$$\boxed{H_{\text{opt}}(z) = \frac{1}{G_s(z)} \left[G_s(z) z^p\right]_+ ,} \qquad (22.117)$$

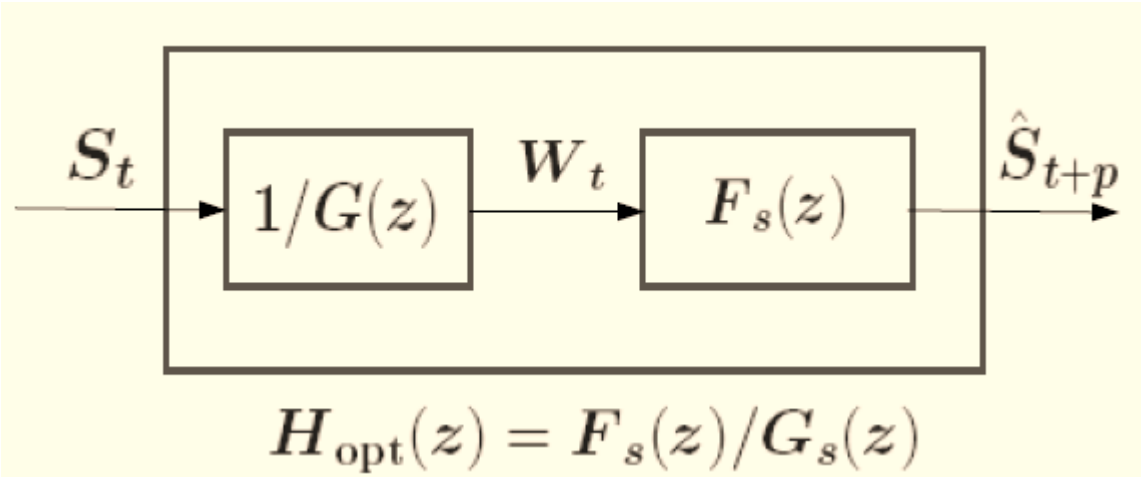


Figure 22.7 An optimal predictor $H_{\text{opt}}(z)$ that acts on the input signal S_t.

$$\boxed{h_{\text{opt}}[k] = \mathcal{Z}^{-1}\left\{\frac{F_s(z)}{G_s(z)}\right\}.}$$

$$\sum_{k=0}^{\infty} |g_s[k]|^2 = E\left[|S_{t+p}|^2\right] = \|S_{t+p}\|^2 \tag{22.119}$$

$$\sum_{k=0}^{\infty} |f_s[k]|^2 = \sum_{k=p}^{\infty} |g_s[k]|^2 = E\left[|\hat{S}_{t+p}|^2\right] = \|\hat{S}_{t+p}\|^2 \tag{22.120}$$

$$\|e_{t+p}\|^2 = \|S_{t+p}\|^2 - \|\hat{S}_{t+p}\|^2 = \sum_{k=0}^{p-1} |g_s[k]|^2 \qquad (22.121)$$

Example 22.3: Pure predictor of an autoregressive signal

$X_t = S_t$

$$P_{ss}(z) = \frac{|A|^2}{(1-\alpha z^{-1})(1-\overline{\alpha} z)}$$

$$P_s(\omega) = \frac{|A|^2}{2\pi\left(1+|\alpha|^2 - 2\Re\{\alpha\}\cos\omega\right)} \qquad (22.122)$$

$$G_s(z) = \frac{A}{1-\alpha z^{-1}}$$

Then, by referring to Figure 22.5, the filtered noise $\{Z_t\}$ and the white noise $\{W_t\}$ are related by

$$Z_t = \alpha Z_{t-1} + A W_t \qquad (22.123)$$

an *autoregressive sequence* of first-order denoted as AR(1)

If the white noise W_t is Gaussian, Z_t is a *Gauss-Markov process* S_t, which is statistically equivalent to Z_t, is also a GMP.

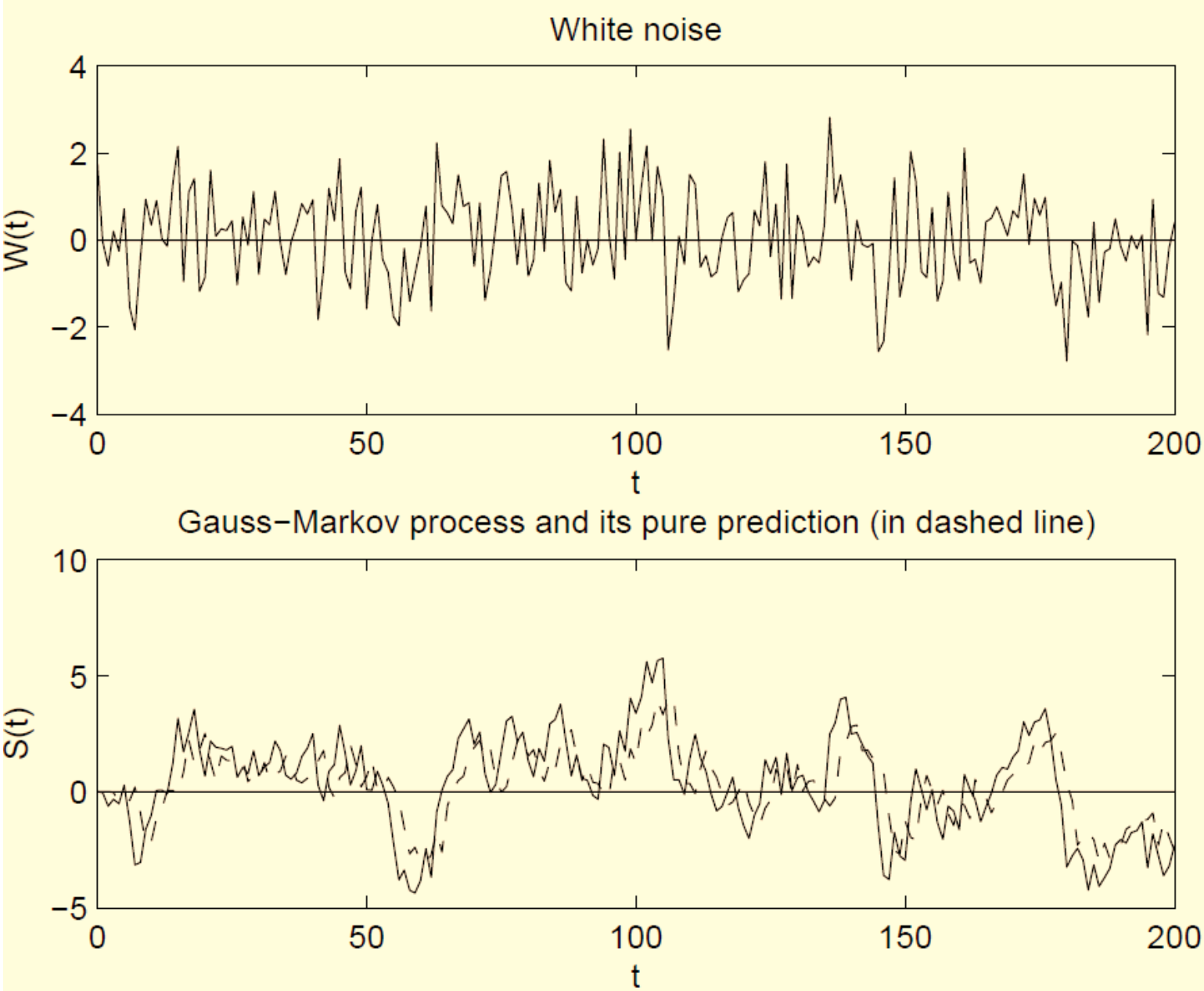
$$g_s[k] = \begin{cases} 0, & k < 0. \\ A\alpha^k, & k = 0, 1, 2, \ldots \end{cases} \tag{22.124}$$

$$f_s[k] = \begin{cases} 0, & k < 0, \\ A\alpha^{k+p}, & k \geq 0. \end{cases}$$

$$F_s(z) = \frac{A\alpha^p}{1 - \alpha z^{-1}}.$$

$$H_{\text{opt}}(z) = \frac{F_s(z)}{G_s(z)} = \alpha^p. \tag{22.125}$$

$$\hat{S}_{t+p} = \alpha^p S_t.$$



The dashed curve is the output of a pure Wiener predictor, which estimates S_t based on S_{t-2}.

The predicted waveform is a scaled (by $\alpha^p = 2^{-1/2}$) and shifted (by $p = 2$ time units) version of the actual signal waveform

$$E[S_{t+p}|S_t] = \rho S_t = \alpha^p S_t, \quad p \geq 0.$$

$$\mathrm{Var}[S_{t+p}|S_t] = \sigma_s^2(1-\rho^2) = \sigma_s^2\left(1-\alpha^{2p}\right)$$

$$|Z_t|^2 = |\alpha Z_{t-1}|^2 + |AW_t|^2 + 2\Re\{\alpha\overline{A}Z_{t-1}\overline{W}_t\}$$

$$\sigma_s^2 = |\alpha|^2\sigma_s^2 + |A|^2.$$

$$\sigma_s^2 = \frac{|A|^2}{1-|\alpha|^2} \qquad (22.128)$$

Theorem 22.4 (Optimality of the best linear pure predictor for a Gaussian process).

If the signal S_t is a Gaussian process, the best linear pure predictor $\hat{S}_{t+p}$ is as good as any predictor, linear or nonlinear, in the minimum mean square error (MMSE) sense.

Proof. Let Y_t be *any* prediction of S_{t+p} based on the input prior to time t, i.e., $\{S_{t'},\ t' < t\}$ Then Y_t is independent of $\{W_k,\ k > t\}$

Since

$$Z_t = Y_t - \hat{S}_{t+p} = Y_t - \sum_{k=-\infty}^{t} g_s[t+p-k]W_k \qquad (22.130)$$

Z_t is also independent of $\{W_k,\ k > t\}$. Furthermore,

$$S_{t+p} - \hat{S}_{t+p} = \sum_{k=t+1}^{t+p} g_s[t+p-k]W_k$$

Z_t and $S_{t+p} - \hat{S}_{t+p}$ are independent, hence orthogonal.

$$
\begin{aligned}
E\left[|S_{t+p}-Y_t|^2\right] &= E\left[\left|S_{t+p}-\hat{S}_{t+p}-Z_t\right|^2\right] \\
&= E\left[\left|S_{t+p}-\hat{S}_{t+p}\right|^2\right] + E\left[|Z_t|^2\right] \\
&\geq E\left[\left|S_{t+p}-\hat{S}_{t+p}\right|^2\right] \qquad (22.131)
\end{aligned}
$$

Hence, the mean square error of any prediction Y_t cannot be made smaller than the error of the best linear prediction $\square$

The above theorem states that if S_t is a Gaussian process, then $E[S_{t+p}|S_k,\ u \leq t] = \sum_{k=0}^{\infty} h[k]S_{t-k}$ with some linear predictor $h[k],\ t \geq 0$.

22.2.5 Prediction of a Random Signal in the Presence of Noise

The first step in solving the general case is to factor out $P_{xx}(z)$:

$$\boxed{P_{xx}(z) = G_x(z)G_x^*(z^{-1}),} \tag{22.132}$$

where $G_x(z)$ is a causal filter, i.e., $g_x[k] = \mathcal{Z}^{-1}\{G_x(z)\} = 0,\ k < 0.$

Example 22.4: Factorization of $P_{xx}(z)$.

Assume that both signal and noise are AR(1), i.e.,

$$S_t = \alpha S_{t-1} + AW_t,\ \ N_t = \beta N_{t-1} + BW_t', \tag{22.133}$$

where W_t and W_t' are independent white noise (not necessarily Gaussian).

In Figure 22.9 we plot S_t, N_t and $X_t = S_t + N_t$, by assuming W_t and W_t' are white Gaussian noise $\sim N(0,1)$, and

$$|A|^2 = |B|^2 = 1,\ \ \alpha = 2^{-1/4} \approx 0.8409,\ \ \beta = 2^{-1} = 0.5.$$

Let us assume that both signal and noise are both AR(1), i.e.,

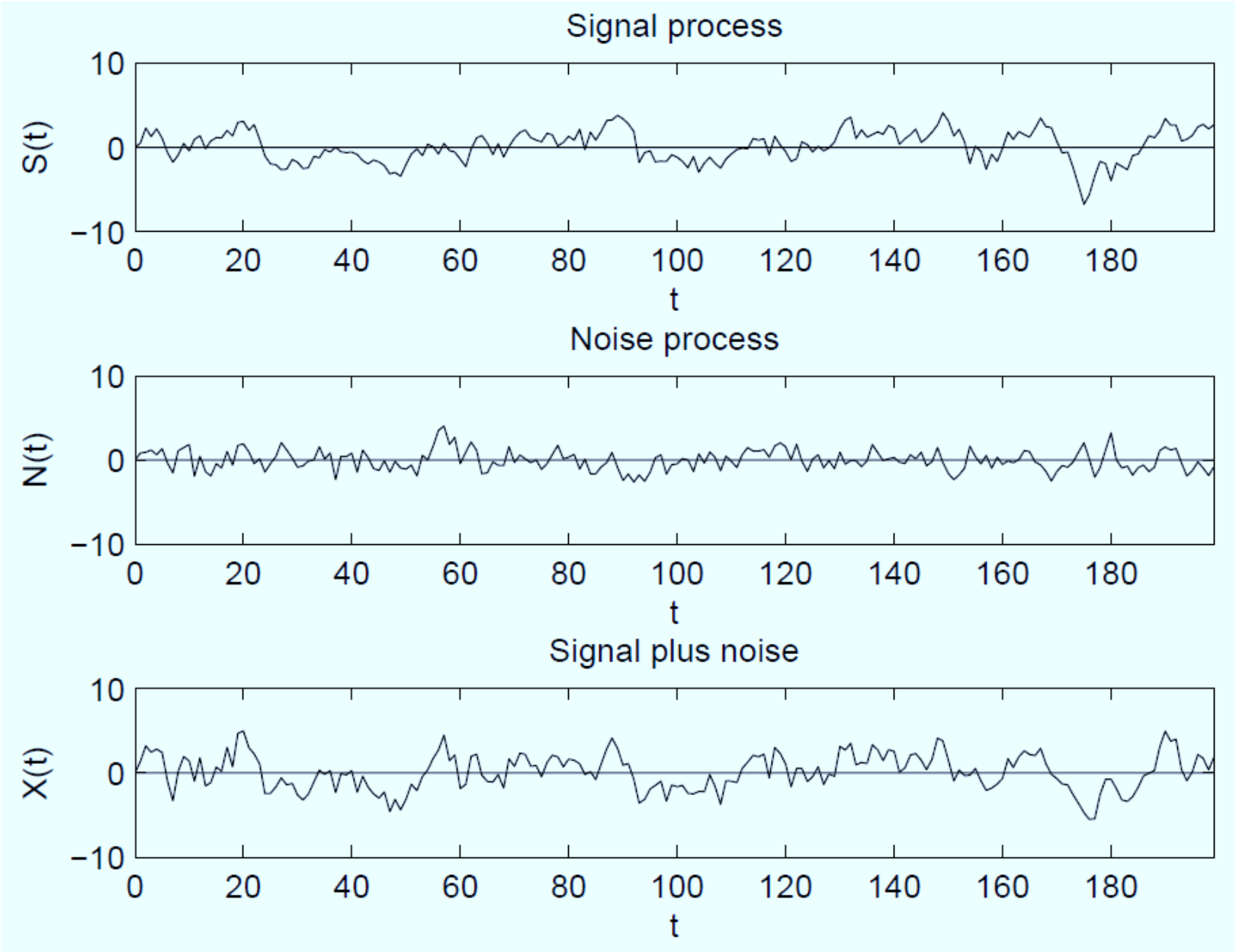


Figure 22.9 Gauss-Markov signal S_t and noise N_t and their superposition

Using (22.128), the signal-to-noise-ratio (SNR) is given by

$$\mathrm{SNR} = \frac{\sigma_s^2}{\sigma_n^2} = \frac{|A|^2(1-|\beta|^2)}{|B|^2(1-|\alpha|^2)} = 2.56 \approx 4.08 \text{ dB}$$

The power spectrums take the following form:

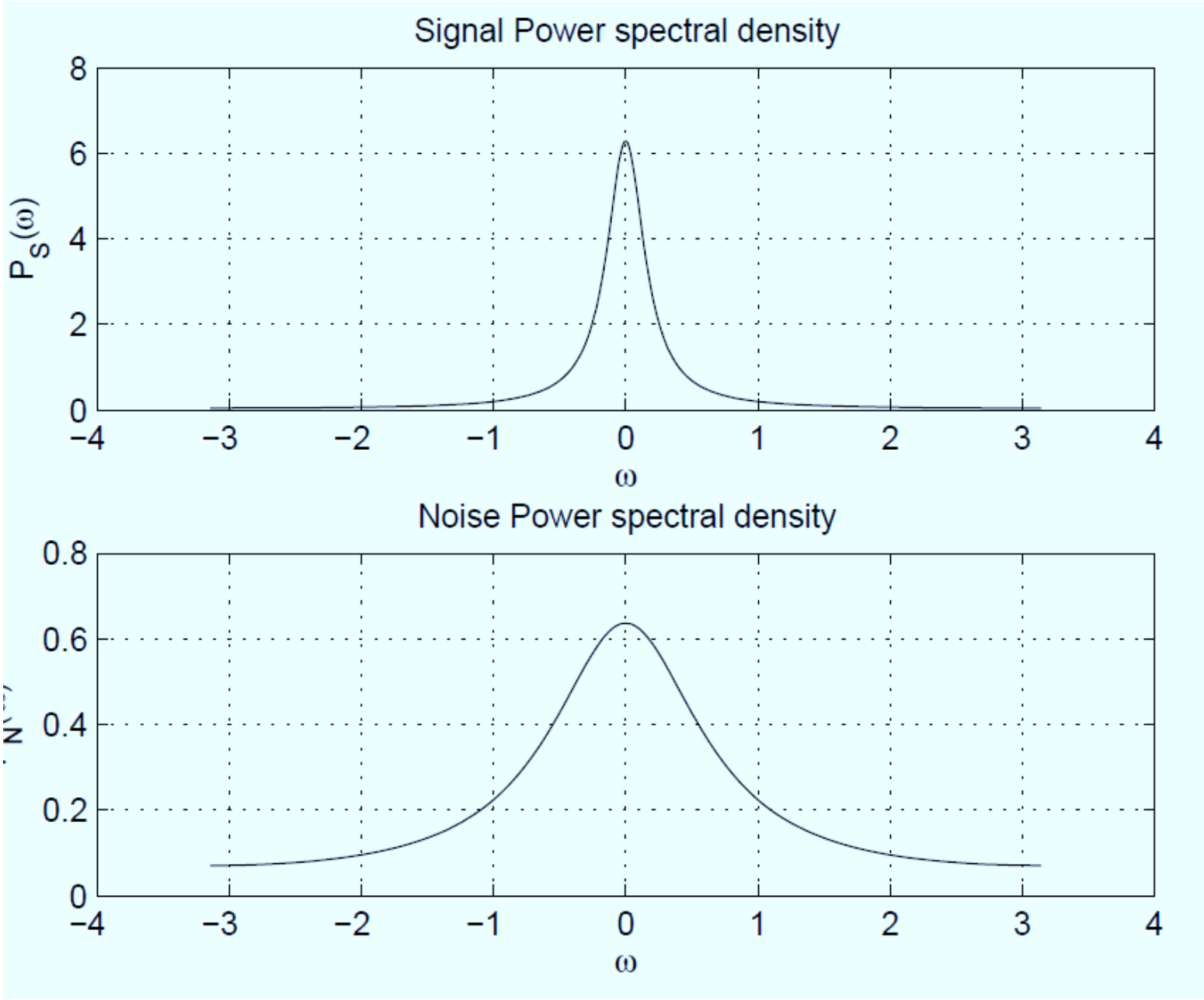
$$P_{ss}(z) = \frac{|A|^2}{(1-\alpha z^{-1})(1-\overline{\alpha} z)}, \quad P_{nn}(z) = \frac{|B|^2}{(1-\beta z^{-1})(1-\overline{\beta} z)} \tag{22.135}$$

The power spectral density

$$P_s(\omega) = \frac{1}{2\pi} P_{ss}(e^{i\omega}) = \frac{|A|^2}{2\pi|1-\alpha e^{-i\omega}|^2}$$

$$= \frac{1}{2\pi(1-2\Re\{\alpha\}\cos\omega + |\alpha|^2)}, \qquad -\pi \le \omega \le \pi,$$

and a similar expression for the noise process.



$$P_{xx}(z) = P_{ss}(z) + P_{nn}(z)$$

$$= \frac{|B|^2(1-\alpha z^{-1})(1-\overline{\alpha}z) + |A|^2(1-\beta z^{-1})(1-\overline{\beta}z)}{(1-\alpha z^{-1})(1-\overline{\alpha}z)(1-\beta z^{-1})(1-\overline{\beta}z)}$$

$$= \frac{|C|^2(1-\gamma z^{-1})(1-\overline{\gamma}z)}{(1-\alpha z^{-1})(1-\overline{\alpha}z)(1-\beta z^{-1})(1-\overline{\beta}z)}$$

where $|C|$ and γ are determined by

$$|C|^2(1+|\gamma|^2) = |B|^2(1+|\alpha|^2) + |A|^2(1+|\beta|^2)$$

$$|B|^2\alpha + |A|^2\beta = |C|^2\gamma.$$

$$\gamma = 0.5175, \text{ and } |C|^2 = 2.3326$$

We can factor $P_{xx}(z)$:

$$P_{xx}(z) = G_x(z)G_x^*(z^{-1}) \tag{22.136}$$

$$G_x(z) = \frac{C(1-\gamma z^{-1})}{(1-\alpha z^{-1})(1-\beta z^{-1})}$$

The inverse Z-transform gives

$$g_x[k] = \frac{C(\alpha-\gamma)}{\alpha-\beta}\alpha^k + \frac{C(\gamma-\beta)}{\alpha-\beta}\beta^k, \quad k = 0, 1, 2, \ldots$$

If we choose C to be real and positive, we find $C = 1.5273$, and

$$g_x[k] = 1.4489 \times 2^{-\frac{k}{4}} + 0.0784 \times 2^{-k}, \quad k = 0, 1, 2, \ldots$$

Clearly, the second term dies down four times as fast as the first term as time k progresses. $\square$

Note that all zeros ($z = 0, \gamma$) and poles ($z = \alpha, \beta$) of $G_x(z)$ are within the unit circle ($|z| = 1$), so that the inverse filter $1/G_x(z)$ should be also physically realizable. Such a linear system is called a *minimum-phase system*.

Let us denote the inverse Z-transform of $G_x^*(z^{-1})$ as $g_x^{(-)}[k]$:

$$g_x^{(-)}[k] = \mathcal{Z}^{-1}\left\{G_x^*(z^{-1})\right\} = \overline{\mathcal{Z}^{-1}\left\{G_x(z^{-1})\right\}}$$
$$= \overline{g_x[-k]} = \overline{g}_x[-k]$$

Therefore,

$$g_x^{(-)}[k] = 0, \quad k > 0 \tag{22.137}$$

referred to as an *anti-causal filter*.

Since $P_{xx}(z) = G_x(z)G_x^*(z)$,

$$R_{xx}[k] = g_x[k] \circledast g_x^{(-)}[k]$$
$$= \sum_{j=-\infty}^{0} g_x[k-j]g_x^{(-)}[j]. \tag{22.138}$$

Define $A(z)$ such that

$$\boxed{P_{sx}(z) = A(z)G_x^*(z^{-1}).} \tag{22.139}$$

By taking the inverse Z-transform,

$$R_{sx}[k] = a[k] \circledast g_x^{(-)}[k] = \sum_{j=-\infty}^{0} a[k-j]g_x^{(-)}[j] \tag{22.140}$$

where

$$\boxed{a[k] = \mathcal{Z}^{-1}\{A(z)\}.} \tag{22.141}$$

By substituting (22.138) and (22.140) into (22.89)

$$\boxed{\sum_{k=0}^{\infty} h[k]R_{xx}[j-k] = R_{sx}[j+p], \quad \text{for all } j \geq 0.} \tag{22.89}$$

$$\sum_{j=-\infty}^{0} a[t+p-j]g_x^{(-)}[j] = \sum_{k=0}^{\infty} h[k] \sum_{j=-\infty}^{0} g_x[t-k-j]g_x^{(-)}[j], \quad t \geq 0, \tag{22.142}$$

or

$$\sum_{j=-\infty}^{0} g_x^{(-)}[j] \left[a[t+p-j] - \sum_{k=0}^{\infty} h[k]g_x[t-k-j] \right] = 0, \quad t \geq 0 \tag{22.143}$$

The last equation is satisfied if

$$a[t+p-j] = \sum_{k=0}^{\infty} h[k]g_x[t-k-j], \quad t \geq 0, \; j < 0 \tag{22.144}$$

$$\boxed{a[t'+p] = \sum_{k=0}^{\infty} h[k]g_x[t'-k], \quad t' > 0.} \tag{22.145}$$

The Z-transform of LHS of (22.145)

$$\begin{aligned}\text{LHS} &= \mathcal{Z}\{a[t'+p]u[t']\} = [A(z)z^p]_+ \\ &= \left[\frac{P_{sx}(z)z^p}{G_x^*(z^{-1})}\right]_+ \triangleq F_{sx}(z). \end{aligned} \tag{22.146}$$

The RHS of (22.145) is the convolution of the two causal functions

$$\text{RHS} = \mathcal{Z}\left\{\sum_{k=0}^{\infty} h[k]g_x[t-k]\right\} = H(z)G_x(z) \tag{22.147}$$

From the last two equations we find

$$\boxed{H_{\text{opt}}(z) = \frac{F_{sx}(z)}{G_x(z)} = \frac{1}{G_x(z)}\left[\frac{P_{sx}(z)z^p}{G_x^*(z^{-1})}\right]_+ .} \tag{22.148}$$

Pure prediction: $N_t = 0$ for all t

$$P_{sx}(z) = P_{ss}(z) = G_s(z)G_s^*(z^{-1})$$

Thus, (22.148) becomes

$$H_{\text{opt}}(z) = \frac{1}{G_s(z)}\left[\frac{P_{ss}(z)z^p}{G_s^*(z^{-1})}\right]_+ = \frac{1}{G_s(z)}\left[G_s(z)z^p\right]_+ \qquad (22.149)$$

which is (22.117).

Smoothing: $p = -d \leq 0$.

An optimal smoothing filter with delay d is found by setting $p = -d$ in (22.148).

Uncorrelated signal and noise:

$$P_{sx}(z) = P_{ss}(z).$$

Then, an optimum p-step predictor is

$$H_{\text{opt}}(z) = \frac{1}{G_x(z)} \left[\frac{P_{ss}(z) z^p}{G_x^*(z^{-1})} \right]_+ \tag{22.150}$$

Similarly, an optimum smoothing filter is given by setting $p = -d \leq 0$.

Example 22.5: Optimal predicting filter for uncorrelated signal and noise

Let us consider an optimal prediction problein Example 22.4. In that example, we found

$$P_{xx}(z) = G_x(z) G_x^*(z^{-1})$$

where

$$G_x(z) = \frac{C(1 - \gamma z^{-1})}{(1 - \alpha z^{-1})(1 - \beta z^{-1})}$$

$$G_x^*(z^{-1}) = \frac{\overline{C}(1 - \overline{\gamma} z)}{(1 - \overline{\alpha} z)(1 - \overline{\beta} z)}$$

$$
\begin{aligned}
\frac{P_{ss}(z)z^p}{G_x^*(z^{-1})} &= \frac{|A|^2}{\overline{C}} \frac{(1-\overline{\beta}z)z^p}{(1-\alpha z^{-1})(1-\overline{\gamma}z)} \\
&= \frac{|A|^2}{\overline{C}} \frac{z^{p+1}(1-\overline{\beta}z)}{(z-\alpha)(1-\overline{\gamma}z)} \\
&= \frac{|A|^2}{\overline{C}} \left[\frac{az^p}{1-\alpha z^{-1}} + \frac{bz^{p+1}}{1-\overline{\gamma}z} \right] \qquad (22.151)
\end{aligned}
$$

where

$$
a = \frac{1-\alpha\overline{\beta}}{1-\alpha\overline{\gamma}}, \quad \text{and} \quad b = \frac{\overline{\gamma}-\overline{\beta}}{1-\alpha\overline{\gamma}}
$$

If $p \geq 1$, the first term in the bracket [ ] of (22.151) contributes to the causal part, whereas the second term is all non-causal

For $p \geq 0$, we find

$$\left[\frac{P_{ss}(z)z^p}{G_x^*(z^{-1})}\right]_+ = \frac{a|A|^2}{\overline{C}}\left[\frac{z^p}{1-\alpha z^{-1}}\right]_+ = \frac{a|A|^2}{\overline{C}}\sum_{k=0}^{\infty}\alpha^k z^{-(k-p)}u_{k-p}$$

$$= \frac{a|A|^2\alpha^p}{\overline{C}}\sum_{t=0}^{\infty}\alpha^t z^{-t} = \frac{a|A|^2\alpha^p}{\overline{C}}\frac{1}{1-\alpha z^{-1}} \qquad (22.152)$$

Thus, from (22.150)

$$\boxed{H_{\text{opt}}(z) = c\,\alpha^p\,\frac{1-\beta z^{-1}}{1-\gamma z^{-1}},} \qquad (22.153)$$

where

$$c = \frac{a|A|^2}{|C|^2} = \frac{(1-\alpha\overline{\beta})|A|^2}{(1-\alpha\overline{\gamma})|C|^2}.$$

By taking the Z-transform, we obtain

$$h_{\text{opt}}[k] = \begin{cases} c\alpha^p, & k = 0; \\ c(\gamma - \beta)\alpha^p \gamma^{k-1}, & k = 1, 2, \ldots \end{cases}$$

which is an almost geometrically decaying function, i.e., $h_{\text{opt}}[k+1]/h_{\text{opt}}[k] = \gamma < 1$ for all $k \geq 1$, but $h_{\text{opt}}[1]/h_{\text{opt}}[0] = \gamma - \beta < \gamma < 1$.

In Figure 22.11 we plot the GMP signal S_t and noisy version $X_t = S_t + N_t$ and the filtered output Y_t, where we set $p = 0$.

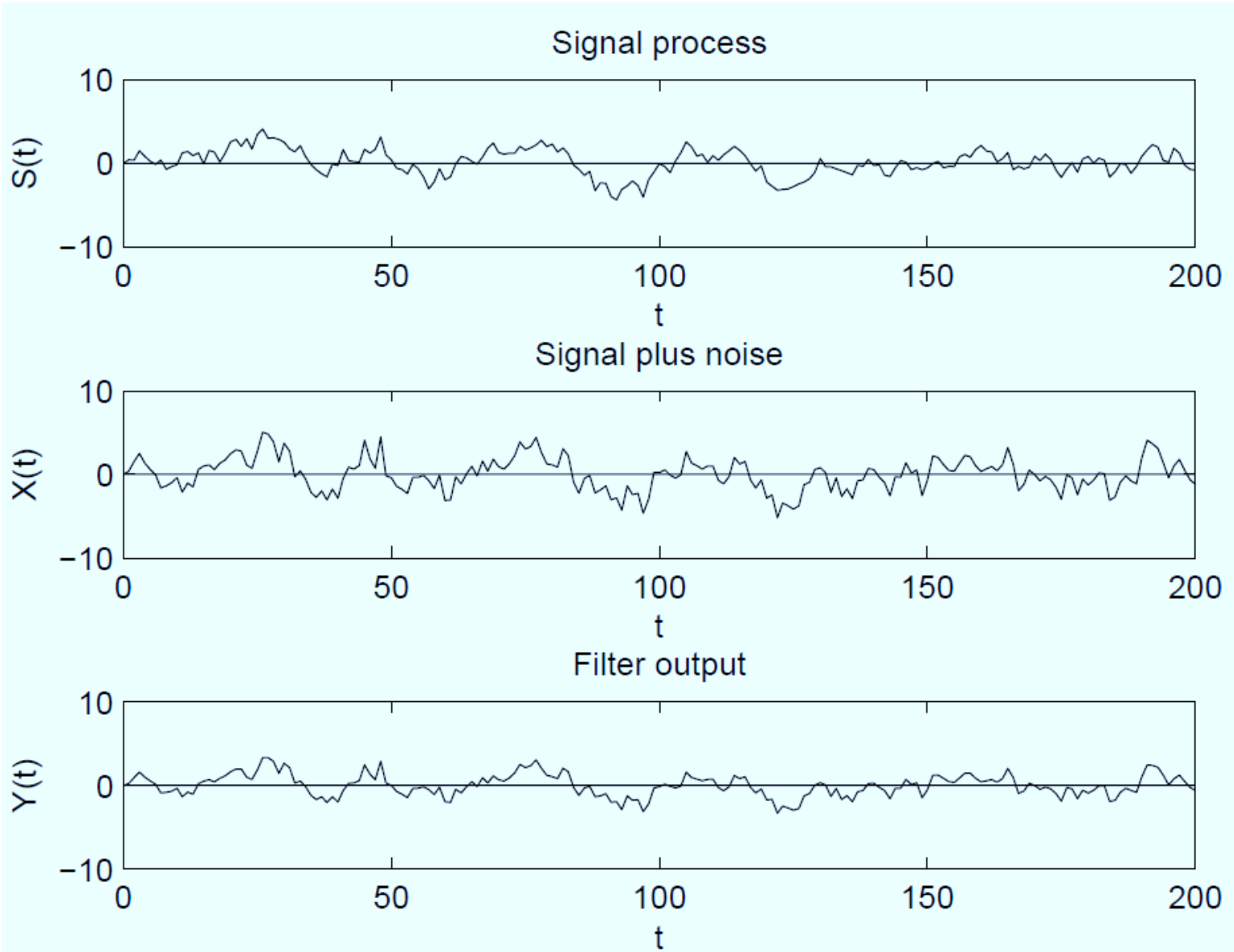


Figure 22.11 The GMP signal S_t, its noisy version X_t, and the filter output $Y_t = \hat{S}_t$ with $p = 0$.

Let us examine the above solution for two special cases.

White noise: $\beta = 0$, then (22.135) becomes $P_{nn}(z) = B$

The predicting filter is found from (22.153)

$$H_{\text{opt}}(z) = \frac{c\alpha^p}{1 - \gamma z^{-1}}$$

Thus,

$$h_{\text{opt}}[k] = \begin{cases} c\,\alpha^p \gamma^k, & k = 0, 1, 2, \ldots \\ 0, & k < 0, \end{cases} \qquad (22.155)$$

Pure prediction: By setting $B = 0$, we have

$$P_{xx}(z) = P_{ss}(z) = \frac{|A|^2}{(1 - \alpha z^{-1})(1 - \overline{\alpha} z)}$$

$$G_s(z) = \frac{A}{1 - \alpha z^{-1}}, \quad G_s^*(z^{-1}) = \frac{\overline{A}}{1 - \overline{\alpha} z}$$

Thus, from (22.149)

$$H_{\text{opt}}(z) = (1 - \alpha z^{-1}) \left[\frac{z^p}{1 - \alpha z^{-1}} \right]_+$$

Following the derivation step in (22.152),

$$\left[\frac{z^p}{1 - \alpha z^{-1}} \right]_+ = \frac{\alpha^p}{1 - \alpha z^{-1}}$$

Thus,

$$H_{\text{opt}}(z) = (1 - \alpha z^{-1}) \frac{\alpha^p}{1 - \alpha z^{-1}} = \alpha^p$$

an "attenuator" by the factor α^p, as obtained in (22.125)

Evaluation of minimum mean square errors:

$$\mathcal{E}_{\min} = E\left[\left|S_{t+p} - \hat{S}_{t+p}\right|^2\right] \tag{22.156}$$

By substituting

$$\hat{S}_t = \sum_{k=0}^{\infty} h_{\text{opt}}[k] X_{t-k} \tag{22.157}$$

$$\mathcal{E}_{\min} = R_{ss}[0] - 2\Re\left\{\sum_{j=0}^{\infty} R_{sx}[j+p] h^*_{\text{opt}}[j]\right\}$$

$$+ \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} h_{\text{opt}}[k] R_{xx}[j-k] h^*_{\text{opt}}[j] \tag{22.158}$$

By substituting

$$R_{sx}[j+p] = \sum_{k=0}^{\infty} h_{\text{opt}}[k] R_{xx}[j-k] \tag{22.159}$$

into (22.158) we have

$$\mathcal{E}_{\min} = R_{ss}[0] - \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} h_{\text{opt}}[k] R_{xx}[j-k] h^*_{\text{opt}}[j] \qquad (22.160)$$

Using Parseval's formula, we can write the above (Problem 22.21)

$$\mathcal{E}_{\min} = \frac{1}{2\pi i}\oint P_{ss}(z)\frac{dz}{z} - \frac{1}{2\pi i}\oint H_{\text{opt}}(z) P_{xx}(z) H^*_{\text{opt}}(z^{-1})\frac{dz}{z} \qquad (22.161)$$

By substituting (22.148) into the above and using $G_x(z)G_x^*(z^{-1}) = P_{xx}(z)$, we finally obtain

$$\boxed{\mathcal{E}_{\min} = \frac{1}{2\pi i}\oint P_{ss}(z)\frac{dz}{z} - \frac{1}{2\pi i}\oint F_{sx}(z)\overline{F}_{sx}(z^{-1})\frac{dz}{z},} \qquad (22.162)$$

where $F_{sx}(z)$ was defined in (22.146):

$$\boxed{F_{sx}(z) = \left[\frac{P_{sx}(z)z^p}{G_x^*(z^{-1})}\right]_+ .} \tag{22.163}$$

Pure prediction case:

$$P_{sx}(z) = P_{xx}(z) = P_{ss}(z) = G_s(z)G_s^*(z^{-1})$$

$$g_s[t] = 0, \quad t < 0$$

Then

$$\mathcal{E}_{\min} = \oint G_s(z)G_s^*(z^{-1})\frac{dz}{z} - \oint F_s(z)\overline{F}_s(z^{-1})\frac{dz}{z} \tag{22.164}$$

$F_s(z)$ is optimum predictor for the whitened signal as in Figure 22.6:

$$F_s(z) = [G_s(z)z^p]_+$$

Using the inverse formula $\mathcal{Z}^{-1}\{G_s(f)z^p\} = g_s[k+p]$,

$$f_s[k] = \mathcal{Z}^{-1}\{F_s(z)\} = g_s[k+p]u[k]$$

Applying the Parseval's formula again,

$$\boxed{\mathcal{E}_{\min} = \sum_{k=0}^{\infty} |g_s[k]|^2 - \sum_{k=0}^{\infty} |f_s[k]|^2 = \sum_{k=0}^{p-1} |g_s[k]|^2.} \qquad (22.165)$$

In referring to Figure 22.6 (a), the MMSE of the pure prediction is given by the norm square of the noncausal part of the second impulse response $g_s[g+p]$.

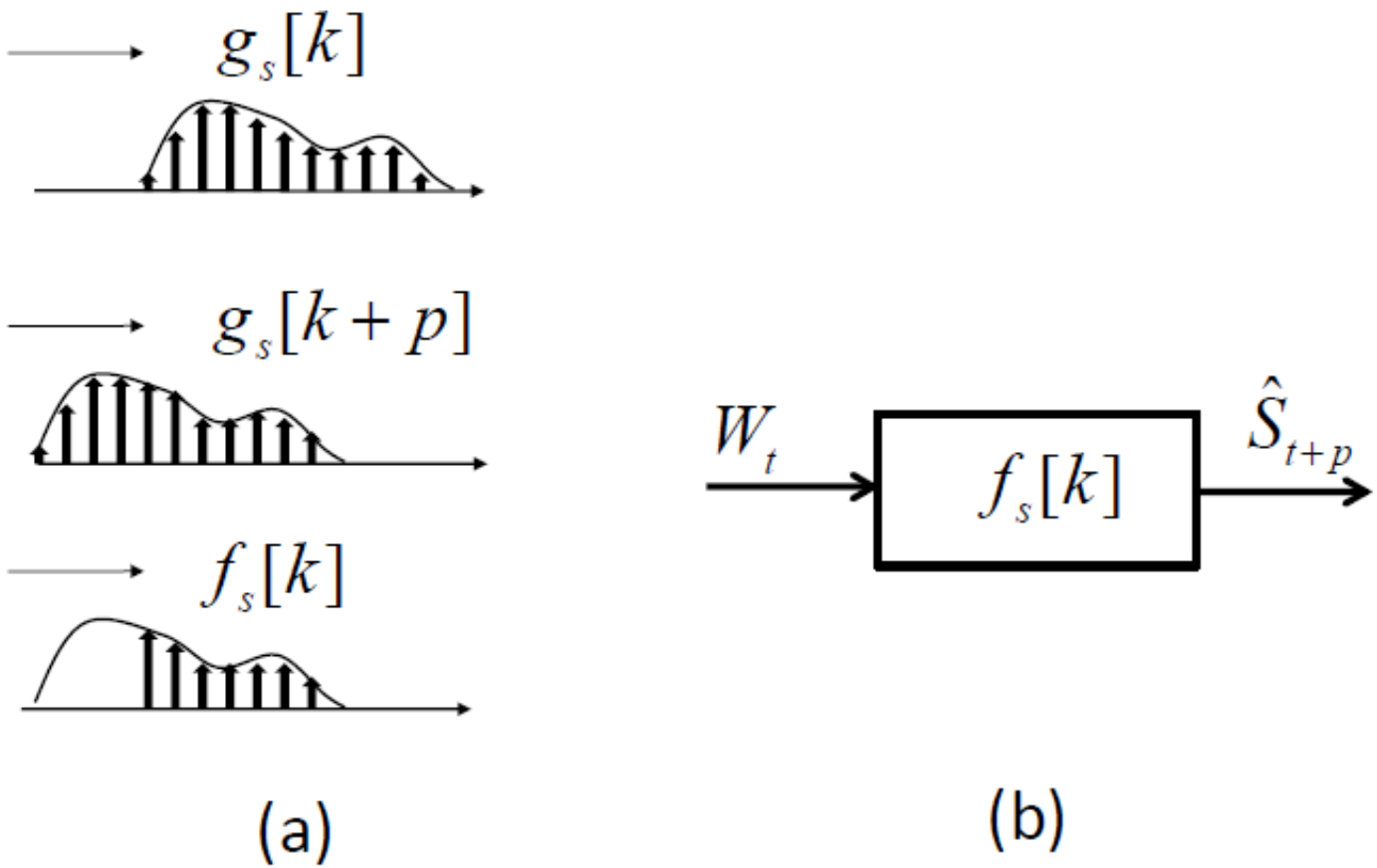


Figure 22.6 (a) The relation between $g_s[k]$ and $f_s[k]$; (b) An optimal predictor $f_s[k] = g_s[k+p]$ that acts on white noise.

Lecture 22:
Maximum-Likelihood Estimation

ELE 525: Random Processes in Information Systems

Hisashi Kobayashi

Department of Electrical Engineering
Princeton University
December 4, 2013

Estimation Theory: Preliminary

18.1 Parameter Estimation

We consider random variable (RV) $\boldsymbol{X}$ with probability distributions $F(\boldsymbol{x};\boldsymbol{\theta})$ with parameter $\boldsymbol{\theta}$,

$$\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_M).$$

The value of parameter $\boldsymbol{\theta}$ is unknown and we want to estimate it from observations

$$\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n) \in \mathcal{X}$$

drawn from the distribution $F(\boldsymbol{x};\boldsymbol{\theta})$. We want to find a function $T(\cdot)$ such that $\hat{\boldsymbol{\theta}} = \boldsymbol{T}(\boldsymbol{x})$ is as close to $\boldsymbol{\theta}$ as possible.

Definition 18.1 (Unbiasedness, efficiency and consistency of an estimator).

1. *An estimator* $\hat{\boldsymbol{\theta}}(\boldsymbol{X})$ *is said to be* ***unbiased****, if* $E[\hat{\boldsymbol{\theta}}(\boldsymbol{X})] = \boldsymbol{\theta}$*; otherwise, it is called* ***biased****. The bias is defined as*

$$b(\boldsymbol{\theta}) = E[\hat{\boldsymbol{\theta}}(\boldsymbol{X})] - \boldsymbol{\theta}. \tag{18.1}$$

2. *An unbiased estimator* $\hat{\boldsymbol{\theta}}^*(\boldsymbol{X})$ *is said to be* ***efficient****, if it is a* ***minimum-variance estimator****, that is,* $\mathbf{Var}[\hat{\boldsymbol{\theta}}^*(\boldsymbol{X})] \leq \mathbf{Var}[\hat{\boldsymbol{\theta}}(\boldsymbol{X})]$ *for any other unbiased estimator* $\hat{\boldsymbol{\theta}}(\boldsymbol{X})$.
3. A sequence of estimators *is said to be* ***consistent****, if the sequence converges in probability to* $\boldsymbol{\theta}$.□

Definition 18.2 (Sufficient statistic). *A statistic* $\boldsymbol{T}(\boldsymbol{X})$ *is said to be* ***sufficient*** *for parameter* $\boldsymbol{\theta}$*, if the conditional probability density (or mass) function of* $\boldsymbol{X}$*, given* $\boldsymbol{T}(\boldsymbol{X}) = \boldsymbol{t}$*, does not depend on* $\boldsymbol{\theta}$. □

This means that, given $\boldsymbol{T}(\boldsymbol{x}) = \boldsymbol{t}$, full knowledge of the measurement $\boldsymbol{x}$ does not bring any additional information concerning $\boldsymbol{\theta}$.

18.1.2 Maximum-Likelihood Estimation

$$\boxed{L_{\boldsymbol{x}}(\boldsymbol{\theta}) \triangleq \begin{cases} f_{\boldsymbol{X}}(\boldsymbol{x};\boldsymbol{\theta}), & \text{for continous RV } \boldsymbol{X}, \\ p_{\boldsymbol{X}}(\boldsymbol{x};\boldsymbol{\theta}), & \text{for discrete RV } \boldsymbol{X}. \end{cases}} \tag{18.24}$$

is called the **likelihood function**: Any value of $\boldsymbol{\theta}$ that maximizes the likelihood function is called a **maximum-likelihood estimate** (MLE),

$$\boxed{\hat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}} L_{\boldsymbol{x}}(\boldsymbol{\theta}).} \tag{18.25}$$

The procedure to find an MLE is called **maximum-likelihood estimation**.

If the likelihood function is differentiable with respect to its parameter, a necessary condition for an MLE to satisfy is

$$\nabla_{\boldsymbol{\theta}} L_{\boldsymbol{x}}(\boldsymbol{\theta}) = \mathbf{0}, \quad \text{i.e.,} \quad \frac{\partial L_{\boldsymbol{x}}(\boldsymbol{\theta})}{\partial \theta_m} = 0, \quad m = 1, 2, \ldots, M. \tag{18.26}$$