Lecture 17:
Continuous-Time Markov Chains (CTMC)

ELE 525: Random Processes in Information Systems

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16.2 Continuous-time Markov Chain (CTMC)

Consider an SMP $X(t)$ for which the sojourn time distribution $F_{ij}(t)$ is

$$F_{ij}(t) = 1 - \exp(-\nu_i t), \ t \geq 0, \ i, j \in S.$$  \hspace{1cm} (16.17)

Note that $F_{ij}(t)$ depends only on $i$, and not on $j$.

It then follows from (16.8) that the distribution function of the sojourn time in state $i$ is given by $F_{S_i}(t) = F_{ij}(t)$ for all $i, j \in S$.

Then $X(t)$ can be shown to be a Markov process and hence is called a CTMC.
16.2.1 Infinitesimal Generator and Embedded Markov Chain of a CTMC

Definition 16.2 (Transition probability matrix function (TPMF)).

For a CTMC $X(t)$, the conditional probabilities

$$P_{ij}(t) = P[X(s + t) = j \mid X(s) = i], \quad i, j \in S, \quad 0 \leq t < \infty,$$

are called transition probability functions, and

$$P(t) = [P_{ij}(t)]$$

is called the transition probability matrix function (TPMF).

The Chapman-Kolmogorov equation (15.20) takes the following form:

$$P_{ik}(s + t) = \sum_{j \in S} P_{ij}(s)P_{jk}(t), \quad s, t \geq 0, \quad i, k \in S. \quad (16.20)$$

$$P(s + t) = P(s)P(t). \quad (16.21)$$
Definition 16.3 (Infinitesimal generator matrix).

The infinitesimal generator (matrix) is defined by

\[ Q = \left. \frac{dP(t)}{dt} \right|_{t=0}. \]  \hspace{1cm} (16.22)

Alternatively,

\[ Q = [Q_{ij}], \]

\[ Q_{ij} = \lim_{h \to 0} \frac{P[X(t + h) = j \mid X(t) = i]}{h} \]

\[ \lim_{h \to 0} \frac{P_{ij}(h)}{h}, \text{ for } i \neq j, \] \hspace{1cm} (16.23)

\[ Q_{ii} = - \sum_{j \neq i} Q_{ij} = \lim_{h \to 0} \frac{P_{ii}(h) - 1}{h}. \] \hspace{1cm} (16.24)
Theorem 16.2 (Stationary distribution of an ergodic CTMC in terms of its EMC).

Suppose \( X(t) \) is an ergodic CTMC with infinitesimal generator matrix \( Q \). Let \( \tilde{\pi} \) denote the stationary distribution of the associated EMC \( \{X_n\} \), i.e., \( \tilde{\pi} \) is the solution to (16.9).

\[
\tilde{\pi}^\top = \tilde{\pi}^\top \tilde{P} \quad \text{and} \quad \tilde{\pi}^\top \mathbf{1} = 1, \tag{16.9}
\]

Then \( X(t) \) has a unique stationary distribution \( \pi \) given by (16.25).

\[
\pi_i = \frac{\tilde{\pi}_i / \nu_i}{\sum_{j \in S} \tilde{\pi}_j / \nu_j}, \quad i \in S, \tag{16.25}
\]

where \( \{\nu_i, i \in S\} \) is a set of transition rates.
16.2.2 Kolmogorov’s Forward and Backward Equations

If we set \( s = h \) in (16.20),

\[
P_{ik}(t + h) = \sum_{j \neq k} P_{ij}(t)P_{jk}(h) + P_{ik}(t)P_{kk}(h).
\]

\[
\frac{P_{ik}(t + h) - P_{ik}(t)}{h} = \sum_{j \neq k} P_{ij}(t)\frac{P_{jk}(h)}{h} + \frac{P_{kk}(h) - 1}{h} P_{ik}(t).
\]

\[
\frac{dP_{ik}(t)}{dt} = \sum_{j \neq k} P_{ij}(t)Q_{jk} + P_{ik}(t)Q_{kk} = \sum_{j \in S} P_{ij}(t)Q_{jk}
\]

known as **Kolmogorov’s forward (differential) equation.**
The solution

\[ P(t) = e^{Qt}, \quad (16.35) \]

If we set \( t = h \) in (16.20),

\[ P_{ik}(h + u) = \sum_{j \neq i} P_{ij}(h)P_{jk}(u) + P_{ii}(h)P_{ik}(u). \]

\[ \frac{dP(t)}{dt} = QP(t), \quad (16.38) \]

known as Kolmogorov’s backward (differential) equation.

If all entries of \( Q \) are bounded, then \( Q \) is said to be uniform and

\[ P(t) = e^{Qt} = \sum_{n=0}^{\infty} \frac{Q^{n}t^{n}}{n!}, \quad t \geq 0. \quad (16.39) \]
\[ p^\top(t) = p^\top(0)P(t), \quad t \geq 0. \]

\( p(t) \) does not depend on \( t \) if and only if \( p(0) = \pi \), where \( \pi \) is the invariant or stationary distribution that satisfies

\[ \pi^\top Q = 0^\top, \quad (16.41) \]

called the global balance equation

\[ \sum_{j \in S} \pi_j Q_{ji} = 0, \quad \text{for all } i \in S, \quad (16.42) \]

which, using property (16.24), can be rewritten

\[ \sum_{j \neq i} \pi_j Q_{ji} = \pi_i \left( \sum_{j \neq i} Q_{ij} \right), \quad \text{for all } i \in S. \quad (16.43) \]
Theorem 16.3 (Stationary distribution of an ergodic CTMC).
An ergodic CTMC with infinitesimal generator matrix $Q$ has a unique stationary distribution $\pi$ satisfying the global balance equation (16.41).

16.2.3 Spectral expansion of the infinitesimal generator

$$Q = U \Gamma U^{-1} = U \Gamma V = \sum_{i \in S} \gamma_i E_i$$  \hspace{1cm} (16.44)

$\Gamma = \text{diag}[\gamma_0, \gamma_1, \gamma_2, \ldots]$, and

$$\det|Q - \gamma_i I| = 0, \ i \in S.$$

The $i$th column vector $u_i$ of $U = [u_i; \ i \in S]$ is the right eigenvector Similarly, the $i$th row vector $v_i$ of $V = U^{-1}$ is the left eigenvector

$$E_i = u_i v_i^\top, \ i \in S,$$

$$v_i^\top u_j = \delta_{ij}, \ i, j \in S.$$
Then the TPM $P(t) = e^{Qt}$ has the spectral expansion

$$P(t) = U \Lambda(t) V = \sum_{i \in S} \lambda_i(t) E_i$$

where $\lambda_i(t)$ is the $i$th entry of the diagonal matrix $\Lambda(t)$

$$\lambda_i(t) = e^{\gamma_i t} \tag{16.45}$$

The state distribution $p(t)$

$$p(t)^\top = p(0)^\top P(t) = p(0)^\top \sum_{i \in S} e^{\gamma_i t} u_i v_i^\top = \sum_{i \in S} a_i e^{\gamma_i t} v_i^\top \tag{16.46}$$

$$a_i = p(0)^\top u_i, \ i \in S.$$ 

the eigenvalue $\gamma_0 = 0$ \hspace{1cm} $u_0^\top = (1, 1, 1, \ldots, 1)$. \hspace{1cm} $a_0 = 1.$

$$p(t) = v_0 + \sum_{i \in S \setminus \{0\}} a_i e^{\gamma_i t} v_i \tag{16.48}$$
The maximum negative (smallest in magnitude) eigen value, \( \gamma_1 \) (i.e., \( \gamma_n < \gamma_1 < \gamma_0 = 0 \) for all \( j \geq 2 \)), determines the rate of convergence. That is, for sufficiently large \( t \),

\[
p(t) \approx v_0 + a_1 e^{\gamma_1} v_1, \quad \text{for large } t. \quad (16.49)
\]

\[
\lim_{t \to \infty} p(t) = v_0,
\]

therefore, it must hold that

\[
v_0 = \pi \quad (16.51)
\]
16.3 Reversible Markov Chains

A continuous-time random process \( X(t) \) is said to be reversible if \( X(t) \) and \( X(\tau - t) \) are statistically identical, for any real value \( \tau \). Note that a reversible process is necessarily stationary.

Similarly, a discrete-time random process \( X_n \) is reversible if it is statistically indistinguishable from its reversed process \( \{X_{-n}\} \).

16.3.1 Reversible DTMC

Consider an ergodic DTMC \( X_n \) defined on a state space \( S \).

**Theorem 16.4 (Reversed balance equations for a DTMC).**

The reversed process \( \tilde{X}_n \) is an ergodic DTMC with the same stationary distribution \( \pi \) and its TPM \( \tilde{P} = [\tilde{P}_{ij}] \) satisfies the following reversed balance equations:

\[
\pi_i \tilde{P}_{ij} = \pi_j P_{ji}, \quad i, j \in S.
\] (16.57)
Proof. To show that the reversed process $\tilde{X}_n$ is a Markov chain, we need to establish that

$$P[\tilde{X}_n = x_0 \mid \tilde{X}_{n-1} = x_1, \ldots, \tilde{X}_{n-m} = x_m]$$

$$= P[\tilde{X}_n = x_0 \mid \tilde{X}_{n-1} = x_1]. \tag{16.58}$$

for any integer $m \geq 1$ and $x_0, x_1, \ldots, x_m \in S$.

LHS of (16.58) can be expressed as

$$P[\tilde{X}_n = x_0, \tilde{X}_{n-1} = x_1, \ldots, \tilde{X}_{n-m} = x_m]$$

$$= \frac{P[\tilde{X}_{n-1} = x_1, \ldots, \tilde{X}_{n-m} = x_m]}{P[\tilde{X}_{n-1} = x_1, \ldots, \tilde{X}_{n-m} = x_m]}$$

$$= \frac{P[X_{-n} = x_0, X_{-n+1} = x_1, \ldots, X_{-n+m} = x_m]}{P[X_{-n+1} = x_1, \ldots, X_{-n+m} = x_m]}$$

$$(a) \quad \frac{\pi_{x_0} P_{x_0 x_1} P_{x_1 x_2} \cdots P_{x_{m-1} x_m}}{\pi_{x_1} P_{x_1 x_2} \cdots P_{x_{m-1} x_m}} = \frac{\pi_{x_0} P_{x_0 x_1}}{\pi_{x_1}}. \tag{16.59}$$

where step (a) follows because $X_n$ is a homogeneous DTMC.
Similarly, the RHS of (16.58) can be expressed as

\[
\frac{P[\tilde{X}_n = x_0, \tilde{X}_{n-1} = x_1]}{P[\tilde{X}_{n-1} = x_1]} = \frac{\pi_{x_0} P_{x_0 x_1}}{\pi_{x_1}}
\]  

(16.60)

Letting \( x_0 = i \) and \( x_1 = j \), (16.60) leads to the reversed balance equations (16.57).

From (16.57) \( \tilde{P}_{x_1 x_0} \), of \( \{\tilde{X}_n\} \) are proportional to \( P_{x_0 x_1} \), of \( \{X_n\} \). Since \( X_n \) is an ergodic DTMC, so too must be \( \tilde{X}_n \).

Next, consider the \( j \)th component of the vector \( \pi^\top \tilde{P} \)

\[
[\pi^\top \tilde{P}]_j = \sum_{i \in S} \pi_i \tilde{P}_{ij} \overset{(a)}{=} \sum_{i \in S} \pi_j P_{ji} \overset{(b)}{=} \pi_j
\]

(16.61)

where (a) follows from the reversed balance equations (16.57) and (b) follows because \( P \) is a stochastic matrix.

Hence, \( \pi^\top \tilde{P} = \pi^\top \), so \( \pi \) is the stationary distribution of \( \tilde{X}_n \). □
Theorem 16.5 (Converse of reversed balance equations for DTMC). 

Let $X_n$ be an ergodic DTMC with TPM $P$. If we can find a TPM $\tilde{P}$ and a probability distribution $\pi = [\pi_i]$, $i \in S$, such that reversed balance equations (16.57) hold. Then $\tilde{P}$ is the TPM of the reversed process $\tilde{X}_n = \{X_{-n}\}$ and $\pi$ is the stationary distribution of both $X_n$ and $\tilde{X}_n$.

Proof. Problem 16.12

Note that the reversed balance equations (16.57) hold whether or not $X_n$ is reversible. Reversibility of $X_n$ further requires

$$P_{ij} = \tilde{P}_{ij}, \quad i, j \in S$$

(16.62)

Substituting (16.62) into (16.57), we obtain the following result.
Theorem 16.6 (Detailed balance equations for DTMC).

An ergodic DTMC $X_n$ is reversible if and only if the following detailed balance equations are satisfied:

$$\pi_i P_{ij} = \pi_j P_{ji}, \quad i, j \in S.$$  \hfill (16.63)

Theorems 16.5 and 16.6 imply

Corollary 16.1 (Converse of detailed balance equations for DTMC).

Let $X_n$ be an ergodic DTMC with TPM $P$. If we can find a probability distribution $\pi = [\pi_i], i \in S$, such that detailed balance equations (16.63) hold, then $X_n$ is reversible and $\pi$ is its stationary distribution.

Theorem 16.7 (Kolmogorov’s criterion for reversibility of a DTMC).

An ergodic DTMC $X_n$ is reversible if and only if

$$P_{x_1x_2} P_{x_2x_3} \cdots P_{x_{n-1}x_n} P_{x_nx_1} = P_{x_1x_n} P_{x_nx_{n-1}} \cdots P_{x_3x_2} P_{x_2x_1}$$

for any sequence of states $x_1, x_2, \ldots, x_n$ in the state space $S$. 